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Proceedings

Foreword

We are very grateful that the *12th International Conference on Computability and Complexity in Analysis* (CCA 2015) has found an accommodating, convenient, and generous host at Meiji University in Tokyo: thanks to the help of the Organizing Committee consisting of Naohi Eguchi, Kojiro Higuchi, Kenshi Miyabe, Ryuhei Mori and Hideki Tsuiki. Moreover, financial support is gladly acknowledged: from the MEXT Project "Exploring the Limits of Computation" (ELC), from Deutsche Gesellschaft der JSPS-Stipendiaten e.V. (German JSPS Club), from JSPS Kakenhi Grant-in-Aid 26700001 (Theory and applications of computational complexity in the continuous world), and from the Kayamori Foundation of Informational Science Advancement.

A continuous series of annual scientific congresses since 1995 – and for the second time in Japan – CCA has grown and evolved into an established and vibrant community, as reflected by the 4 invited talks and by 22 extended abstracts (plus several informal talks) originating in 13 different countries that were accepted for presentation. Indeed the Programme Committee has done a terrific job in reviewing, commenting, recommending, and choosing among the many submissions: Many thanks to Andrej Bauer, Stephen A. Cook, Guido Gherardi, Daniel Graça, Hajime Ishihara, Ker-I Ko, Timothy McNicholl, André Nies, and Mariko Yasugi!

CCA originates in Recursive Analysis with focus on qualitative computability questions about real numbers, functions, and subsets. This continues to constitute the core of our field with contributions for example by Laurent Bienvenu, Santiago Figueira, Daniel Graça, Rutger Kuyper, Benoit Monin, Arno Pauly, Alexander Shen, Shu-Ming Sun, and Ning Zhong. At the same time our scope has significantly expanded over the last decades, exhibiting and pursing connections to several other areas. For instance the topological aspects of computing over continuous universes motivate investigations by, among others, Matthew de Brecht, Cameron Freer, Vassilios Gregoriades, Robert Kenny, Kei Matsumoto, Arno Pauly, Matthias Schröder, and Kazushige Terui. The computational content of mathematical theorems is regularly explored in settings such as Weihrauch degrees or Constructive/Reverse Mathematics: see for instance Alexander Kreuzer's invited talk as well as investigations by, say, Vasco Brattka, Makoto Fujiwara, Benedikt Löwe, Hugo Nobrega, Arno Pauly, Tahina Rakotoniaina, and Kazuto Yoshimura. Complexity-theoretic refinements of computability results are presented for example by Hugo Férée, Tomohiro Katayama, Amaury Pouly, and Florian Steinberg. Indeed we believe Real Complexity Theory to be a very promising direction: providing a resource-oriented algorithmic foundation to numerical calculations, it bridges from Recursive Analysis to practical aspects of real computation! These, and questions of programming languages for computing over the reals, are subjects of invited talks by Sicun Gao, Vikram Sharma, and Kohei Suenaga as well as of contributions from Ulrich Berger, Christine Gaßner, Kenji Miyamoto, Eike Neumann, Norbert Preining, Helmut Schwichtenberg, Hideki Tsuiki, and Pedro Francisco Valencia Vizcaíno. In fact we are proud and glad that four renowned scholars have agreed to organize with us a dedicated *German-Japanese Workshop on Theory and Practice of Real Computation* on July 12 co-located with CCA2015. We are grateful for the generous support of German Research and Innovation Forum Tokyo (DWIH Tokyo) that made this workshop possible.

So please enjoy this conference, enjoy its rich and scintillating variety of invited, contributed, and informal talks, enjoy the opportunity for discussions and (initiating) collaborations – and enjoy your stay in the world's most populous city!

Martin Ziegler (Programme Committee Chair) and
Akitoshi Kawamura (Organizing Committee Chair)

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ON THE UNIFORM COMPUTATIONAL CONTENT OF COMPUTABILITY THEORY

VASCO BRATTKA, MATTHEW HENDTLASS, AND ALEXANDER P. KREUZER*

We demonstrate that the Weihrauch lattice can be used to study the uniform computational content of computability theoretic properties and theorems in one common setting. The properties that we study include diagonal non-computability, hyperimmunity, complete extensions of Peano arithmetic, 1-genericity, Martin-Löf randomness and cohesiveness. The theorems that we include in our case study are the Low Basis Theorem of Jockusch and Soare, the Kleene-Post Theorem and Friedman's Jump Inversion Theorem. It turns out that all the aforementioned properties and many theorems in computability theory, including all theorems that claim the existence of some Turing degree, have very little uniform computational content. They are all located outside of the upper cone of binary choice (also known as LLPO) and we call problems with this property *indiscriminative*. Since practically all theorems from classical analysis whose computational content has been classified are discriminative, our observation could yield an explanation for why theorems and results in computability theory typically have very little direct consequences in other disciplines such as analysis.

Two notable exceptions to this are the Low Basis Theorem which is discriminative, this is perhaps why it is considered to be one of the most applicable theorems in computability theory, and the Baire category theorem (or to be precise certain formulations of it) which is an indiscriminative principle occurring in mathematical analysis. We will see that the Baire category theorem is related to 1-genericity.

In some cases a bridge between the indiscriminative world and the discriminative world of classical mathematics can be established via a suitable residual operation and we demonstrate this in case of the cohesiveness problem, which turns out to be the quotient of two discriminative problems, namely the limit operation and the jump of Weak Kőnig's Lemma.

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*Presenting author.

Faculty of Computer Science, Universität der Bundeswehr München, Germany and Department of Mathematics & Applied Mathematics, University of Cape Town, South Africa[1]
E-mail address: `Vasco.Brattka@cca-net.de`

University of Canterbury, Christchurch, New Zealand
E-mail address: `Matthew.Hendtlass@canterbury.ac.nz`

Department of Mathematics, National University of Singapore, Singapore[2]
E-mail address: `matkaps@nus.edu.sg`

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Computable Analysis from an Engineering Perspective

Sicun Gao

MIT

Cyber-Physical Systems (CPS), such as airplanes, nuclear plants, and cardiac pacemakers, tightly integrate computational and physical components. While these systems are ubiquitous and highly safety-critical, we do not have a methodological foundation for building them in reliable, optimized, and secure ways. I will survey our work in the search for such a foundation, and demonstrate how results and methods from computable analysis are applied in various contexts. In particular, I will outline a framework that aims to accomplish the following:

– Understand the computational complexity of controlling nonlinear and hybrid dynamical systems (e.g. stabilizing a helicopter to hover). Such problems are commonly considered as highly undecidable in traditional computation models. I will show that reasonable upper bounds can be obtained in the delta-decisions framework, as direct applications of complexity results from computable analysis.
– Enhance automation in the design and implementation of control systems through automated reasoning engines. These engines are designed to match the complexity of the problems to be solved (generally NP-hard). The key is to engineer exponential algorithms to behave well in practice, by combining the full power of logical reasoning and numerical algorithms. I introduce our solver dReal and show some promising experimental results.
– Certify correctness of control software through formal proofs. All algorithms that are used for the design and analysis of these systems should produce mathematical proofs that can be machine-checked. This requires a thorough formal analysis of control theory and numerical computing.

I wish to highlight the importance of computable analysis in connecting logical, computational, and engineering methods. I believe the unification of these approaches will bring design automation and reliability to an unprecedented level in the broad field of engineering.

Real Numbers In Exact Geometric Computation

Vikram Sharma

The Exact Geometric Computation (EGC) approach to handle non-robustness in computational geometry. This approach emphasizes guaranteeing the correct geometry even when numerical errors occur in computation. The EGC paradigm [Yap97] has been very successful at handling the issues of non-robustness in geometric algorithms. The major software libraries `LEDA`[LED95] and `CGAL`[CGA98] are based upon this paradigm. I will give an overview of the (real) computability model underlying the EGC approach and the challenges in implementing it. There are many interesting questions here, one of them being extending the EGC approach to non-algebraic computations, i.e., computations that involve both polynomials and functions such as exponentiation, logarithms etc.

A fundamental problem that arises in implementing this approach is Real Root Isolation: given a polynomial in one variable with real numbers as coefficients, find an enclosing interval (with rational endpoints) for every real root of the polynomial such that distinct roots are assigned disjoint intervals. A frequently used algorithm for solving this problem is a subdivision approach combined with Descartes's rule of signs [CA76]. I will describe the algorithm and its analysis based on the framework of continuous amortization [BKY09]. Subsequently, I will describe recent improvements in the theory and practice of these algorithms that attain nearly optimal running times [SB15]. I will also consider the more general problem of isolating roots of analytic functions. The challenge here is to devise guaranteed unconditional algorithms, i.e., algorithms that do not make any non-degeneracy assumptions on the input. I will describe some of our recent work in this direction based on the theory of soft-predicates [SSY13].

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Nonstandard Analysis Meets Programming Language Theory

Kohei Suenaga

Kyoto University

We recently proposed *nonstandard programming languages* [3, 6, 7], languages that incorporate an *infinitesimal number* (i.e., a positive number less than any positive real number) as a constant. The infinitesimal number, constructed using nonstandard analysis [5], enables modeling of continuous and hybrid dynamics without using differential equations. One novel feature is that a program analysis for (standard) languages can be transferred to the nonstandard languages. Hence, various static verification techniques including program logic and abstract interpretation can be applied to continuous and hybrid systems without much change. This talk presents an essence of nonstandard program language, recent result, several ongoing work, and future direction.

Nonstandard programs

Example 1. Let us start from an imperative nonstandard program written in While^{dt} [6]. The following program c_s models simple harmonic motion of a mass point:

$$(x, v, t) := (x_0, 0, 0); \textbf{while } (t < a)\{(x, v, t) := (x + v \cdot dt, v - k \cdot x \cdot dt, t + dt);\}.$$

Here, t is the clock, x_0 is the initial position, k is the spring constant, and dt denotes an infinitesimal value. At each iteration of the loop body, c_s changes the value of each variable infinitesimally following differential equation $x'' = -kx$; this change is expressed using dt.

In order to formally define the behavior of c_s, we need to address the following issues.

- Denotation of dt: Since an infinitesimal value does not exist in the set of reals, we need to construct the value in a mathematically sound way.
- Denotation of **while** loop: The loop in c_s brings the value of t from 0 to a value not less than 1; thus, a reasonable semantics should express $\lceil \frac{a}{dt} \rceil$ times of iterations of the loop body, which is in general "infinitely" many times.

The key to address these issues is *nonstandard analysis* proposed by Robinson. Nonstandard analysis constructs a set $^*\mathbb{R}$ called *hyperreals* so that it satisfies the axioms of real numbers and so that it contains exotic numbers that can be interpreted as infinitesimals and infinities. By using the idea of nonstandard analysis, we define the semantics of the nonstandard languages in a reasonable way [6, 7]. For example, with our semantics, we can prove that the value of x at the end of the execution of c_s is infinitely close to the position of the mass point at time a.

Verifying nonstandard programs

One main theorem of nonstandard analysis is *transfer principle* [2, 5]. It guarantees that a predicate on mathematical structures over $\mathbb{R}$ holds if and only if it holds on (certain subset of) mathematical structures over $^*\mathbb{R}$[1].

By transfer principle, we can transfer static verification techniques for the standard languages to the nonstandard languages. We indeed implemented an automated static analyzer for While^{dt} in the previous paper [4].

[1] We do not present the concrete statement here. See Goldblatt [2] for detail.

Example 2. The previous paper [6] shows that Hoare logic [8, Chap. 6] remains to be sound and relatively complete for While^{dt} without changing inference rules. Thus, by finding a loop invariant of the loop body, we can verify various properties of the program.

For c_s, there is[2] a loop invariant $v^2 + kx^2 = k(1+k\,dt^2)^{\frac{t}{dt}} x_0^2$ (notice $(v - kx\,dt)^2 + k(x+v\,dt)^2 = (1+k\,dt^2)(v^2+kx^2)$.) By using transfer principle, we can prove $(1+k\,dt^2)^{\frac{t}{dt}} < 1+\varepsilon$ for any positive reals k, t, and ε, and hence $v^2 + kx^2 < (1+\varepsilon)kx_0^2$ is a loop invariant. From this loop invariant, we can prove $|x| < x_0\sqrt{1+\varepsilon}$.

Ongoing work and future direction

Verification of a nonstandard program requires powerful invariant-synthesis algorithm, especially one that can deal with *nonlinear* formulas. There have been much progress in this area recently (e.g.,[1]); we expect these techniques can be applied to verification of nonstandard programs.

Our current verifier uses an off-the-shelf computer algebra system as a backend decision procedure. Since such procedure is oblivious of the theory of hyperreals, it treats dt as an unknown real constant. The knowledge on the theory is indeed useful as the the example of c_s suggests, where we used $(1 + k\,dt^2)^{\frac{t}{dt}} < 1 + \varepsilon$ based on the fact that dt is an infinitesimal.

In the set of the hyperreals, there are infinitely many infinitesimal numbers. (For example, if ∂ is an infinitesimal number, then $\frac{\partial}{2}$ is also.) Choice of the denotation of dt in general affects the behavior of a While^{dt} program. (For example, program $t := 0; \textbf{while}\ (t \neq 1)\{t := t + dt\}$ terminates if and only if $\frac{1}{dt}$ is a "natural" number.) Good characterization of the set of programs that is robust to the choice of dt is an interesting future direction.

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[2] We derived the loop invariant manually; our current implementation does not automatically infer it.

LOW FUNCTIONS OF REALS

KATRIN TENT AND MARTIN ZIEGLER

We introduce a notion of computable function on $\mathbb{R}^N$ and prove some basic properties. We develop a notion of computable functions on the reals along the lines of the bit-model as described in [2]. In contrast to the algebraic approach towards computation over the reals developed in [1], our approach goes back to Grzegorczyk and the hierarchy of elementary functions and real numbers developed by him and Mazur (see [4] footnote p. 201).We give several applications, among them a short proof of Yoshinaga?s theorem that periods are elementary (they are actually lower elementary). We also show that the lower elementary complex numbers form an algebraically closed field closed under exponentiation and some other special functions. We also report about more recent results concerning differential equations.

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LOGIC FOR GRAY CODE COMPUTATION

ULRICH BERGER AND KENJI MIYAMOTO AND HELMUT SCHWICHTENBERG AND HIDEKI TSUIKI

Gray code (also called the reflected binary code) is widely known in digital communication, due to its property that the Hamming distance between adjacent Gray codes is always 1. Based on Gray code, Di Gianantonio and Tsuiki have studied independently an expansion of real numbers as streams of $\{0, 1, \bot\}$ each of which contains at most one $\bot$ standing for undefinedness. [4, 9]. Tsuiki called it the modified Gray expansion and has also studied computability of real numbers, and has presented several algorithms to do real number computation via Gray code. The motivation of this paper is to shed light on the logical aspect of Gray code computation from the constructive standpoint. We formalize the pre-Gray code in the *Theory of Computable Functionals*, TCF in short, and also in the proof assistant Minlog[1], which is an implementation of TCF, by means of inductive and coinductive definitions [7]. In order to make use of Tsuiki's idea in TCF, we introduce the pre-Gray code which is a modified Gray expansion represented as ordinary streams. Through the realizability interpretation we extract from proofs programs as terms (in an extension T^+ of Gödel's T) involving (higher type) recursion and corecursion operators. The correctness of the extracted programs is automatically ensured by the soundness theorem. As case studies we extract real number algorithms in our setting of the pre-Gray code: translators from signed digit streams into pre-Gray codes and vice versa, the average of two real numbers in the pre-Gray code and a bounded translation from pre-Gray code into Gray code. Within Minlog the extracted programs are executable, and also can be exported to Haskell. The corresponding Minlog proof script[2] is available in the official Minlog package.

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[1]See `http://www.minlog-system.de/`

[2]See `examples/analysis/gray.scm` in the home directory of Minlog.

Related work. There are programming languages which can process modified Gray expansion directly. Di Gianantonio used the parallel if operator `pif` to access $1\bot$-sequences [4]. Tsuiki and Sugihara study an extension of Haskell with the non-deterministic choice operator `gamb` which works as McCarthy's `amb` operator [11]. Tsuiki studies a logic programming language with guarded clauses and committed choice [10]. Terayama and Tsuiki study an extension of PCF with parallelism [8]. In this paper we avoid using the above features by adopting the pre-Gray code. Concerning stream based real arithmetic, signed digit streams are used to study real number computation by Wiedmer [12, 13]. Its corecursive treatment is studied by Ciaffaglione and Di Gianantonio in Coq [3]. Berger and Seisenberger study program extraction to obtain programs dealing with signed digit streams [1]. Some of their results are formalized by Miyamoto and Schwichtenberg in TCF and Minlog [5, 6]. Chuang studies the average and the multiplication of real numbers using coinduction in Agda via the Curry-Howard isomorphism [2].

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ON THE COMPUTATIONAL CONTENT OF RAMSEY'S THEOREM

VASCO BRATTKA AND TAHINA RAKOTONIAINA

Abstract. We present results on the classification of the computational content of Ramsey's Theorem in the Weihrauch lattice. The computational content of Ramsey's Theorem has been intensively studied and the theorem exhibits an unusual discrepancy between its uniform and non-uniform computational content. We present a survey on recent results included in [DDH+15, Dzh15, Hir15, HJ15, Rak15].

1. Summary

In this talk we plan to highlight some important algebraic properties of Ramsey's Theorem. By $\mathsf{RT}_{n,k}$ we denote the multi-valued function that represents Ramsey's Theorem of size n for k colors. If $k = \mathbb{N}$, then the finite number of colors is arbitrary and unspecified. We obtain the following theorem on products.

Theorem 1.1 (Products). $\mathsf{RT}_{n,\mathbb{N}} \times \mathsf{RT}_{n+1,k} \leq_{\mathrm{sW}} \mathsf{RT}_{n+1,k+1}$ *for all* $n, k \geq 1$.

For one, one can use this theorem to conclude that Ramsey's Theorem has increasing complexity with increasing numbers of colors. On the other hand it also implies the following corollary, which shows that arbitrary products of Ramsey's Theorem of a certain size can be accommodated by Ramsey's Theorem of the next larger size.

Corollary 1.2 (Finite parallelization). $\mathsf{RT}^*_{n,k} \leq_{\mathrm{W}} \mathsf{RT}_{n+1,2}$ *for all* $n, k \geq 1$.

In contrast to this, parallelization requires an increase of the size by 2. That this is sufficient follows from the next theorem and we can also prove that this increase is necessary.

Theorem 1.3 (Delayed Parallelization). $\widehat{\mathsf{RT}_{n,k}} \leq_{\mathrm{sW}} \mathsf{RT}_{n+2,2}$ *for all* $n, k \geq 1$.

From this result we can quite easily derive some important lower bounds of Ramsey's Theorem, which generalize and improve certain results that were recently proved by Hirschfeldt and Jockusch [HJ15]. Here $\mathsf{SRT}_{n,k}$ denotes the restriction of $\mathsf{RT}_{n,k}$ to stable colorings.

Corollary 1.4 (Lower bounds). $\lim \leq_{\mathrm{W}} \mathsf{SRT}_{3,2}$, $\mathsf{WKL}' \leq_{\mathrm{W}} \mathsf{RT}_{3,2}$ *and* $\mathsf{WKL}^{(n)} \leq_{\mathrm{W}} \mathsf{SRT}_{n+2,2}$ *for all* $n \geq 2$.

We will also discuss uniform versions of the relation of Ramsey's Theorem to boundedness and induction principles from reverse mathematics. It turns out that the Weihrauch chain $\mathsf{K}_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{K}'_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{C}'_{\mathbb{N}} \leq_{\mathrm{W}} ...$ can be seen

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as an analogue of the chain $\mathsf{B}\boldsymbol{\Sigma}^0_1 \subseteq \mathsf{I}\boldsymbol{\Sigma}^0_1 \subseteq \mathsf{B}\boldsymbol{\Sigma}^0_2 \subseteq \mathsf{I}\boldsymbol{\Sigma}^0_2 \subseteq$ ... Here $\mathsf{K}_\mathbb{N}, \mathsf{C}_\mathbb{N}$ denote compact and closed choice respectively, whereas $\mathsf{B}\boldsymbol{\Sigma}^0_n$ and $\mathsf{I}\boldsymbol{\Sigma}^0_n$ denote boundedness and induction principles from reverse mathematics [Hir15].

On the positive side, we can for instance prove the following.

Corollary 1.5. $\mathsf{K}''_\mathbb{N} \leq_\mathrm{W} \mathsf{SRT}_{2,\mathbb{N}}$.

Corollary 1.5 can be seen as the uniform version of a theorem of Cholak, Jockusch and Slaman [CJS01], see also [Hir15, Theorem 6.89], which states that $\mathsf{SRT}^2_{<\infty}$ proves $\mathsf{B}\Sigma^0_3$ over RCA_0.

On the negative side we can prove the following result that highlights some difference between the uniform concept of Weihrauch reducibility and reverse mathematics.

Corollary 1.6. $\mathsf{K}'_\mathbb{N} \not\leq_\mathrm{W} \mathsf{SRT}_{2,k}$ *for all* $k \geq 1$.

In view of the fact that $\mathsf{BWT}_\mathbb{N} \equiv_\mathrm{W} \mathsf{RT}_{1,\mathbb{N}}$ this shows that the reverse mathematics result [CJS01] and see [Hir15, Theorem 6.82] that $\mathsf{SRT}_{2,2}$ proves $\mathsf{RT}^1_{<\infty}$ over RCA_0 cannot be proved uniformly (the proof contains a nonconstructive case distinction).

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Faculty of Computer Science, Universität der Bundeswehr München, Germany and Department of Mathematics & Applied Mathematics, University of Cape Town, South Africa[1]

E-mail address: Vasco.Brattka@cca-net.de

Department of Mathematics & Applied Mathematics, University of Cape Town, South Africa[2]

E-mail address: fenoira@gmail.com

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Extending continuous valuations on quasi-Polish spaces to Borel measures

Matthew de Brecht*

National Institute of Information and Communications Technology (NICT), Center for Information and Neural Networks (CiNet), Osaka, Japan

Continuous valuations are a useful alternative to Borel measures for developing a constructive approach to probability and measure theory [5, 3, 4, 8]. Some justification for using valuations in place of measures is given by the numerous valuation extension results (such as [7], [1], and [6]), which show that in many important cases a valuation can be uniquely extended to a Borel measure. Some of the most general results are due to M. Alvarez-Manilla [1], who showed that every locally finite continuous valuation on a regular or locally compact sober space has a unique extension to a Borel measure.

Here we show that every locally finite continuous valuation on a quasi-Polish space [2] uniquely extends to a Borel measure. Our result is not subsumed by M. Alvarez-Manilla's results because of the existence of non-regular non-locally-compact quasi-Polish spaces such as $\mathcal{K}(\omega^\omega)$ (the space of compact saturated subsets of the Baire space with the upper Vietoris topology). On the other hand, our result only applies to countably based spaces, and it does not include non-Polish regular spaces, such as the rationals $\mathbb{Q}$, which are covered by M. Alvarez-Manilla's results.

The main merit of our result is that it is a nice combination of simplicity and generality. Quasi-Polish spaces include many spaces of interest in measure theory, such as Polish spaces and ω-continuous domains, but our proof only requires the valuation extension property for $\mathcal{P}(\omega)$, the natural numbers with the Scott-topology, and a simple preservation lemma for $\mathbf{\Pi}^0_2$-subspaces.

We let $\overline{\mathbb{R}}_+$ denote the space of extended non-negative reals $[0, +\infty]$ with the Scott-topology. Given a topological space X, we let $\mathcal{O}(X)$ denote the lattice of open subsets of X with the Scott-topology.

A *valuation* on a topological space X is a function $\nu\colon \mathcal{O}(X) \to \overline{\mathbb{R}}_+$ satisfying:

1. $\nu(\emptyset) = 0$ (strictness),
2. $\nu(U \cup V) + \nu(U \cap V) = \nu(U) + \nu(V)$ for all opens $U, V \in \mathcal{O}(X)$ (modularity),
3. $U \subseteq V$ implies $\nu(U) \leq \nu(V)$ for all opens $U, V \in \mathcal{O}(X)$ (monotonicity).

A *continuous valuation* on X is a valuation which is continuous with respect to the Scott-topologies on $\mathcal{O}(X)$ and $\overline{\mathbb{R}}_+$. A valuation ν on X is *bounded* or *finite* if $\nu(X) < \infty$, and ν is *locally finite* if for every $x \in X$ there is an open neighborhood U of x such that $\nu(U) < \infty$.

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If X is a countably based space, then every σ-additive measure defined on the Borel subsets of X restricts to a continuous valuation on X. The converse does not hold in general without additional constraints on X. We will say that a topological space X has the *valuation extension property* if every locally finite continuous valuation on X has a unique extension to a σ-additive measure defined on the Borel subsets of X. We restrict attention to locally finite valuations in order to guarantee uniqueness of the extension.

The next lemma generalizes an observation mentioned at the top of Section 4 in [6] concerning the preservation of valuation extension results to G_δ-subspaces. Note that $\mathbf{\Pi}^0_2$-sets are strictly more general than G_δ-sets when working with non-metrizable spaces (see [2], for example).

Lemma 1. *If X is a countably based T_0-space with the valuation extension property, then every $\mathbf{\Pi}^0_2$-subspace of X has the valuation extension property.*

Proof. Assume $Y \in \mathbf{\Pi}^0_2(X)$ and let $\nu\colon \mathcal{O}(Y) \to \overline{\mathbb{R}}_+$ be a locally finite continuous valuation.

As shown by M. Alvarez-Manilla (Lemma 4.1 in [1]), every locally finite continuous valuation on a space X has a unique extension to a Borel measure if and only if every finite continuous valuation on X has a unique extension. Therefore, it suffices to prove our $\mathbf{\Pi}^0_2$-preservation result under the assumption that $\nu(Y) < \infty$. This assumption is necessary because our extension of ν to ν_X in the next paragraph preserves finiteness but may not preserve local finiteness of valuations.

Let $e\colon Y \to X$ be the topological embedding of Y into X. Then the preimage function $e^{-1}\colon \mathcal{O}(X) \to \mathcal{O}(Y)$ is continuous, hence the function $\nu_X\colon \mathcal{O}(X) \to \overline{\mathbb{R}}_+$ defined as $\nu_X = \nu \circ e^{-1}$ is continuous. Clearly, $\nu_X(X) < \infty$, and by using the fact that $\nu_X(U) = \nu(e^{-1}(U)) = \nu(U \cap Y)$ for each $U \in \mathcal{O}(X)$, it is easy to see that ν_X is a finite continuous valuation on X.

Using the valuation extension property of X, the valuation ν_X extends uniquely to a σ-additive measure μ_X on the Borel subsets of X. We let μ denote the restriction of μ_X to the Borel subsets of Y.

Since Y is a Borel subset of X, it is clear that μ is a well defined σ-additive measure on the Borel subsets of Y. Therefore, it only remains to show that μ extends ν. Towards this end, we first prove that $\mu_X(X \setminus Y) = 0$. As $X \setminus Y$ is a $\mathbf{\Sigma}^0_2$-subset of X we can represent it as a countable union

$$X \setminus Y = \bigcup_{i \in \omega} U_i \setminus V_i,$$

with U_i, V_i open subsets of X. Assume for a contradiction that $\mu_X(X \setminus Y) > 0$. The σ-additivity of μ_X implies there is $i \in \omega$ such that $\mu_X(U_i \setminus V_i) > 0$. It follows that

$$\mu_X(U_i \setminus V_i) = \nu_X(U_i) - \nu_X(U_i \cap V_i) = \nu(U_i \cap Y) - \nu(U_i \cap V_i \cap Y),$$

hence $\nu(U_i \cap Y) > \nu(U_i \cap V_i \cap Y)$. On the other hand, $U_i \cap Y \subseteq U_i \cap V_i \cap Y$ follows from the assumption that $U_i \setminus V_i$ is disjoint from Y. This contradicts the monotonicity property of valuations. Therefore, $\mu_X(X \setminus Y) = 0$, and it follows that any Borel subset of X disjoint from Y must also have a μ_X-measure of zero.

To finish the proof, let U be any given open subset of Y and let V be an open subset of X such that $V \cap Y = U$. Then

$$\nu(U) = \nu_X(V) = \mu_X(V) = \mu_X(U) + \mu_X(V \setminus Y) = \mu_X(U) + 0 = \mu(U),$$

hence μ is a σ-additive Borel measure extending ν. □

The valuation extension property for quasi-Polish spaces now follows from the well known fact that $\mathcal{P}(\omega)$ has the valuation extension property [7] and that every quasi-Polish space embeds into $\mathcal{P}(\omega)$ as a $\mathbf{\Pi}^0_2$-subspace [2].

Theorem 1. *Every quasi-Polish space has the valuation extension property.* □

We next show that the preservation result in Lemma 1 is in some sense optimal. Let S_1 denote the natural numbers with the cofinite topology, and let S_D denote the countable chain $(\omega, \leq)$ with the Scott-topology. The continuous valuation defined as $\nu(U) = 1$ if and only if U is non-empty demonstrates that both S_1 and S_D fail to have the valuation extension property. Neither S_1 nor S_D is sober, but they are $\mathbf{\Sigma}^0_2$-subspaces of their sobrifications, which both happen to be quasi-Polish. We have proven the following.

Proposition 1. *The valuation extension property can fail to be preserved by* $\mathbf{\Sigma}^0_2$*-subspaces.* □

It can be shown that every non-sober countably based T_0-space contains a $\mathbf{\Pi}^0_2$-subspace homeomorphic to either S_1 or S_D. We therefore obtain the following necessary condition for the valuation extension property, which was originally proved by M. Alvarez-Manilla [1] in a slightly more general form.

Proposition 2. *Every countably based* T_0*-space with the valuation extension property is sober.* □

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WEAKENINGS OF CAUCHY CONVERGENCE

HANNES DIENER AND ROBERT S. LUBARSKY
DEPT. OF MATHEMATICS AND STATISTICS
UNIVERSITY OF CANTERBURY
CHRISTCHURCH, NEW ZEALAND
HANNES.DIENER@CANTERBURY.AC.NZ
DEPT. OF MATHEMATICAL SCIENCES
FLORIDA ATLANTIC UNIVERSITY
BOCA RATON, FL 33431
ROBERT.LUBARSKY@ALUM.MIT.EDU

The question "what is a real number" has been a long-standing foundational question. The two most standard answers – a Dedekind cut and (an equivalence class of) a Cauchy sequence – go back to the nineteenth century, and are equivalent classically anyway, so often the issue in unproblematic. Constructively, though, those two notions are in general unequal [6]. In fact, in the right axiomatic context, the latter can be a set, while the former a proper class [10].

Intermediate notions between Cauchy and Dedekind reals have been considered by various researchers. For instance, Escardo and Simposon [5] suggested the Cauchy closure of the rationals, the least set containing $\mathbb{Q}$ closed under the process of taking Cauchy sequences. It is not in general true that this is the Cauchy sequences of rationals: there could be a Cauchy sequence of Cauchy sequences of rationals, which is not equivalent to (i.e. does not have the same limit as) any sequence of rationals [7].

In another direction, various authors ([2, 12, 1, 3, 11, 9]) have investigated different kinds of sequentializations of Cauchy convergence, as paradoxical as that might sound, under the names of metastability, local stability, and almost Cauchyness. We investigate these and other related principles, with an eye toward determining the valid implications among them, and, for those implications that do not hold, just what additional principle is needed for the implication. For instance, Tao [12] showed classically that every metastable sequence is Cauchy; we show that that theorem is equivalent to LPO. (If you don't know what this stands for, please find one of the reviewers of this paper, and you might win a beer.)

If time permits, we will mention the notion of a non-trivial distance function to a set, even if there aren't any points in the set [8]. This cannot happen with $\mathbb{R}$, requiring instead higher dimensional spaces.

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THE TOPOLOGY OF UNIVERSAL GRAPHONS — EXTENDED ABSTRACT —

CAMERON FREER AND JAN REIMANN

Over the past decade, graphons have emerged as a major new concept in graph theory (see for example [Lov12] for a comprehensive treatment). Essentially being continuous realizations of (random) graphs that characterize the limits of sequences of finite graphs, they allowed for novel approaches to fundamental graph-theoretic theorems such as Szemerédi's Regularity Lemma, drawing on techniques from analysis and probability theory. Another important impact of graphons was a new understanding of symmetric constructions of *countable* structures, which are also characterized by graphons via a sampling procedure, and especially of universal, homogeneous such structures. For example, Petrov and Vershik [PV10] constructed graphons with the property that a countable sample almost surely yields a (triangle-free) universal, homogeneous graph. The approach was recently generalized and extended to other structures by Ackerman, Freer, and Patel [AFP12].

Among other things, this line of work opened up a novel approach to defining algorithmic randomness for infinite structures. Fouché and Nies [Nie13, §15] define a notion of Martin–Löf randomness for the rational linear ordering $(\mathbb{Q}, <)$ using the Glasner–Weiss measure, the unique invariant measure on its isomorphism class. Recently Ackerman, Freer, Kwiatkowska, and Patel [AFKP14] used constructions in the style of [AFP12] to characterize all countable relational structures for which there is a unique invariant measure, and hence for which there is a canonical notion of algorithmic randomness defined analogously.

However, all of the constructions of [PV10], [AFP12], and [AFKP14] appear to result in considerably more complicated graphons than many of those that typically arise in other applications. In the present work, we examine whether this complexity is necessary, and prove a sense in which any countable structure obtainable in this way must arise from a graphon that is complex in the sense of having noncompact topology.

Definition. A *graphon* is a (Lebesgue) measurable function $W : [0,1]^2 \to [0,1]$ such that $W(x,y) = W(y,x)$ for all $x, y \in [0,1]$.

A graphon can be thought of as an edge-weighted undirected graph on a continuous set of vertices. $W(x,y)$ represents the probability that there is an edge between distinct vertices x and y.

A main feature of graphons is their interpretation as *limits of dense graph sequences.* A sequence (G_n) of finite simple graphs with $|V(G_n)| \to \infty$ *converges* if for every finite graph F, the relative number $t(F, G_n)$ of copies of F in G_n stabilizes. Every convergent sequence (G_n) can be assigned a unique (up to weak isomorphism) graphon W as a limit object so that the continuous density of each F in W (defined via an integral) equals the limit of $t(F, G_n)$. For details see [Lov12].

For example, a sequence of random graphs (G_n), where each edge is chosen to be present independently with probability 1/2 will almost surely converge to the constant graphon

$$W(x,y) \equiv \frac{1}{2}.$$

Given a graphon W, one can sample points $x_1, \dots x_n$ independently and uniformly from $[0,1]$ and form a graph sequence (G_n) by placing edges between i and j independently with probability $W(x_i, x_j)$ for $1 \le i < j \le n$. Then almost surely (G_n) converges to W.

We can also sample countable infinite graphs this way. For example, if we use a countable independent uniform sequence of points $x_1, x_2, x_3, \dots \in [0,1]$ to sample from the constant graphon $W \equiv 1/2$ in this way, we obtain almost surely the *Rado graph*, the unique (up to isomorphism) countable, universal, homogeneous graph.

One may ask now whether other countable universal structures can be realized in a similar way. A prominent example is Henson's universal K_n-free graph ($n \ge 3$) [Hen71]. Here K_n

denotes the complete graph on n vertices. A probabilistic construction of the Henson graph "from below", i.e., via a sequence of finite graphs — as is straightforward for the Rado graph — has long eluded researchers.

Petrov and Vershik [PV10] gave a probabilistic construction "from above", using graphons. Their graphon is considerably more complicated than the "completely random" constant graphon $W \equiv 1/2$. This is due to the fact that the presence of any "grayscale" region (of positive measure) would force the existence of K_n subgraphs with positive probability, and hence the sampled countable graph would not be K_n-free. Therefore, the Petrov-Vershik graphon has the property of being $\{0, 1\}$-valued.

A graphon W is *random-free* if $W(x, y) \in \{0, 1\}$ for every $x, y \in [0, 1]$; in this case one may think of W as coding a measurable graph. For such graphons, we define the *neighbor set* of a vertex $x \in [0, 1]$ as

$$E_x = \{y \in [0, 1] \colon W(x, y) = 1\}.$$

A random-free graphon is called *twin-free* if $E_x \triangle E_y$ has positive (Lebesgue) measure for all $x \neq y$.

Definition (Petrov and Vershik [PV10])**.** A random-free, twin-free graphon W is *countably universal* if for every set of distinct points from $[0, 1]$,

$$x_1, x_2, \ldots, x_n, y_1, \ldots, y_m,$$

the set

$$\bigcap_{i,j} E_{x_i} \cap E^{\complement}_{y_j}$$

has non-empty interior.

Petrov and Vershik define a universal K_n*-free* graphon in a similar way.

Theorem (Petrov and Vershik [PV10])**.** *There exists a universal (K_n-free) countably universal graphon.*

Petrov and Vershik also show that if a sequence $x_1, x_2, x_3, \ldots$ is sampled i.i.d. from $[0, 1]$ with respect to a uniform (or any other positive, non-atomic) distribution from a universal (K_n-free) graphon, the induced substructure one obtains is almost surely an isomorphic copy of the Rado graph, or the K_n-free Henson graph, respectively. (This is a special case of the general sampling procedure described above.)

In this paper, we investigate the complexity of universal graphons from a topological point of view. One can of course consider random-free graphons as subsets of $[0, 1]^2$, but this does not reflect the graph-theoretical properties. Instead, we consider a *neighborhood metric*:

$$r_W(x, y) = \int \big|W(x, z) - W(y, z)\big| \, d\lambda(z),$$

where λ is Lebesgue measure. For random-free twin-free graphons, this is indeed a metric. For general graphons, one has to identify vertices with measure-theoretically identical neighborhoods, but this can be done without changing the underlying graph-theoretical properties. Furthermore, one can always assume that the "purified" space is complete with respect to the r_W metric (for details see, e.g., [Lov12, §13.4]).

Definition. Let W be a graphon, and let $x_0, x_1, x_2, \ldots$ be a sequence in $[0, 1]$. The *Cantor scheme* $\mathcal{C}$ induced by (x_n) is defined as follows:

$$C_\varnothing = [0, 1],$$

and

$$C_{\sigma^\frown 0} = C_\sigma \cap E_{x_{|\sigma|}}, \quad C_{\sigma^\frown 1} = C_\sigma \cap E^{\complement}_{x_{|\sigma|}}.$$

If one can find a Cantor scheme of uniformly distributed measure in W, then W is not compact.

Proposition. *If there exists a sequence* (x_n) *and a* $\delta > 0$ *such that for the induced Cantor scheme* $\mathcal{C}$,

$$\lambda C_\sigma \geq \delta 2^{-n},$$

then W *is not compact in the* r_W*-topology.*

Our main result shows that universal graphons of a certain type cannot be compact. This includes the graphons constructed by Petrov and Vershik [PV10], as well as the constructions of universal graphons in [AFP12]. This indicates that if we want to deterministically construct universal objects, the resulting objects must exhibit a certain complexity.

Definition. A random-free graphon W has *continuous realization of types* if there exists a function

$$\begin{array}{rccc} f: & [0,1]^{<\omega} \times [0,1]^{<\omega} & \to & [0,1]^2 \\ & ((x_1,\dots,x_n),(y_1,\dots,y_m)) & \mapsto & (l,r) \end{array}$$

that satisfies $l < r$ and is continuous almost everywhere, and for all pairs of tuples $(x_1,\dots,x_n)$, $(y_1,\dots,y_m)$,

$$[l,r] \subseteq \bigcap_{i,j} E_{x_i} \cap E_{y_j}^{\complement}.$$

It is not hard to see that a random-free graphon W with continuous realization of types is countably universal.

Theorem. *If a graphon* W *has uniformly continuous realization of types, then it is not compact in the* r_W*-topology.*

The universal (K_n-free) graphons constructed in [AFKP14] also satisfy uniformly continuous realization of types. Hence this theorem also implies that there are continuum-many constructions of each of the Rado graph and Henson K_n-free graphs arising from noncompact graphons, i.e., as the random induced substructure of a Borel graph that is noncompact in this sense.

The main result also has consequences for the box-counting dimension of countably universal graphons.

Corollary. *If a graphon* W *has uniformly continuous realization of types, then the metric space* (W, r_W) *has infinite box-counting (Minkowski) dimension.*

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Massachusetts Institute of Technology
E-mail address: freer@mit.edu

Department of Mathematics, Pennsylvania State University, University Park
E-mail address: jan.reimann@psu.edu

On the Computational Complexity of Positive Linear Functionals on $C[0;1]$*

Hugo Férée[1,2] and Martin Ziegler[1]

[1] Dept. of Mathematics, TU Darmstadt, Schlossgartenstr.7, D-64289 Darmstadt, Germany
[2] Université de Lorraine, LORIA, UMR 7503, Vandœuvre-lès-Nancy, F-54506, France
Inria, Villers-lès-Nancy, F-54600, France
CNRS, LORIA, UMR 7503, Vandœuvre-lès-Nancy, F-54506, France

Abstract. According to the Riesz–Markov–Kakutani Representation Theorem and the Lebesgues Decomposition Theorem, every positive (i.e. monotone) linear functionals on the space $C[0;1]$ of continuous functions on the unit interval is a linear combination of three distinct types, corresponding prototypes being (i) evaluation, (ii) Riemann integration, and (iii) integration w.r.t. the Cantor measure. The first is polynomial-time computable, the second (ii) is known to correspond to the discrete complexity class $\#\mathsf{P}_1$; and we establish the same to also hold for (iii).

Devising a complexity theory of higher-type computation is an ongoing endeavour since at least 25 years [Cook91, KaCo10, FeHo13]. Perhaps more modestly, we are interested in classifying the continuous linear functionals Ψ on the space $C[0;1]$ of continuous functions on the real unit interval: first non-uniformly, that is, investigate the computational complexity of the real number $\Psi(f)$ for arbitrary but fixed polynomial-time computable $f \in C[0;1]$; and then uniformly with (approximations to) f 'given' by means of oracle access, yet still for fixed Ψ.

According to the *Riesz–Markov–Kakutani Representation Theorem* precisely every positive (i.e. monotone) linear functional $\Psi : C[0;1] \to \mathbb{R}$ is of the form $\Psi(f) = \int_0^1 f(t)\, d\nu(t)$ for some regular Borel measure ν on $[0;1]$. *Lebesgue's Decomposition Theorem* in turn asserts each such ν to admit a (unique) decomposition $\nu = \nu_{\mathrm{d}} + \nu_{\mathrm{c}} + \nu_{\mathrm{s}}$, where

i) ν_{d} is discrete:
The 'prototype' of a discrete measure being Dirac's family δ_z with $\delta_z([a;b]) = 1$ if $z \in [a;b]$ and $\delta_z([a;b]) = 0$ otherwise. The induced positive linear functional is simply evaluation at z — and polynomial-time computable uniformly in z. More generally every discrete measure on $[0;1]$ has the form $\nu_{\mathrm{d}} = \sum_{j\in\mathbb{N}} \delta_{z_j} \cdot w_j$ for two sequences $(z_j) \subseteq [0;1]$ and $(w_j) \subseteq [0;\infty)$ with $\sum_j w_j < \infty$.

ii) ν_{c} is absolutely continuous w.r.t. the prototypical Lebesgues measure λ.
The complexity of its induced positive linear functional on $C[0;1]$, namely of definite Riemann integration, has been characterized as $\#\mathsf{P}_1$; and indefinite Riemann integration as $\#\mathsf{P}$ [Ko91, THEOREM 5.32d]. Moreover restricting to continuously differentiable argument does not reduce the worst-case complexity. In general, according to the classical Radon–Nikodym Theorem, to every absolutely continuous measure ν_{c} on $[0;1]$ there exists some measurable $\varphi : [0;1] \to [0;\infty)$ such that $\int_0^x f(t)\, d\nu_{\mathrm{c}}(t) = \int_0^x f(t)\varphi(t)\, dt$ holds for all $f \in C[0;1]$ and $0 \le x \le 1$.

iii) ν_{s} is singular continuous.
The prototype of a singular continuous measure being Cantor's, that is, with cumulative

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distribution given by the Devil's Staircase or Cantor–Lebesgues–Vitali function $S : [0;1] \to [0;1]$, Hölder-continuous with exponent $\alpha := \ln(2)/\ln(3)$, and inducing as functional the parametric Riemann-Stieltjes integral $(f,x) \mapsto \int_0^x f(t)\,dS(t)$.

Recall that Cantor's measure 'lives' on the Cantor set

$$\mathcal{C} = \bigcup\nolimits_{n\in\mathbb{N}} \mathcal{C}_n, \quad \mathcal{C}_n := \bigcup\nolimits_{k=0}^{2^n-1} I_{n,k} \tag{1}$$

having Hausdorff Dimension α, where $I_{n,0},\ldots,I_{n,2^n-1}$ are pairwise disjoint intervals of length 3^{-n} (but receive weight 2^{-n}) each such that $I_{n,k} \subseteq I_{n-1,\lfloor k/2 \rfloor}$. We furthermore presume familiarity with standard discrete complexity classes

$$\mathsf{P} \subseteq \mathsf{NP}_1 \subseteq \mathsf{P}^{\#\mathsf{P}_1} \subseteq \mathsf{P}^{\#\mathsf{P}} \subseteq \mathsf{PSPACE} \subseteq \mathsf{EXP}$$

and their second-order counterparts **P**, **FP**, and **#P**; cmp. [KaCo10].

Our main results are concerned with (iii) and collected in the following

Theorem 1. *a) Cantor's Function $S : [0;1] \to [0;1]$ is computable within polynomial time. Moreover $C[0;1] \ni f \mapsto f \circ S \in C[0;1]$ and $C^1[0;1] \ni f \mapsto f \circ S \in C^{0,\alpha}[0;1] \subsetneq C^1[0;1]$ constitute well-defined second-order polynomial-time computable reductions from definite Riemann to (Hölder-)* continuous*, definite Cantor integration.*

b) There exists a polynomial-time computable smooth *(i.e. infinitely often differentiable) $h : [0;1] \to [0;1]$ such that the definite Cantor integral $\int_0^1 h(t)\,dS(t)$ is not computable in polynomial time unless $\#\mathsf{P}_1 \subseteq \mathsf{FP}$.*

c) Let $f : [0;1] \to [0;1]$ be computable in polynomial time and suppose $\#\mathsf{P}_1 \subseteq \mathsf{FP}$. Then the definite Cantor integral $\int_0^1 f(t)\,dS(t)$ is again computable in polynomial time.

*d) Every $F \in$ **#P** admis a second-order polynomial-time reduction to smooth* in*definite Cantor integration*

$$C^\infty([0;1],[0;1]) \ni f \mapsto \big([0;1]^2 \ni (x,y) \mapsto \int_x^y f(t)\,dS(t) \in [0;1]\big) \in C^{0,\alpha}([0;1],[0;1]) \ .$$

e) Continuous in*definite Cantor integration*

$$C([0;1],[0;1]) \ni f \mapsto \big([0;1]^2 \ni (x,y) \mapsto \int_x^y f(t)\,dS(t) \in [0;1]\big) \in C^{0,\alpha}([0;1],[0;1]) \ ,$$

admits a second-order polynomial-time reduction to **#P**.

In-/definite Cantor integration is thus as hard as in-/definite Riemann-Stieltjes integration: the prototypes of positive linear functionals $\Psi : C[0;1] \to \mathbb{R}$ of types (ii) and (iii).

The question remains as to whether, and in which sense, these prototypes for Riesz–Markov–Kakutani are also representative complexity-wise. It is easy to find Ψ that are harder than these prototypes: for example evaluation (i) at some EXP–complete point $z \in [0;1]$. Absolutely continuous integration is always (either trivial or) $\#\mathsf{P}_1$–hard according to Item b) in the following

Proposition 2. *a) There exist polynomial-time (in both j and the output precision) computable sequences $x_j \searrow 0$ and $w_j \searrow 0$ such that the functional $\Lambda : C[0;1] \ni f \mapsto \sum_j w_j \cdot f(x_j) \in \mathbb{R}$ is well-defined with $\Lambda(\mathrm{id}) = 1$; yet there exists a polynomial-time computable f such that $\Lambda(f)$ is polynomial-time computable iff $\mathsf{FP} \supseteq \#\mathsf{P}_1$.*

b) Let $g : [0;1] \to \mathbb{R}$ *be polynomial time computable and non-identically zero. Then there exists a polynomial-time computable* $f \in C^\infty[0;1]$ *such that* $\int_0^1 f(t) \cdot g(t)\, dt$ *is not polynomial-time computable unless* $\mathsf{FP} \supseteq \#\mathsf{P}_1$.

c) Modify the Cantor set in Equation (1) to $\tilde{\mathcal{C}} := \bigcap_{n\in\mathbb{N}} \bigcup_{k=0}^{2^n-1} \tilde{I}_{n,k}$ *where* $\tilde{I}_{n,k}$ *now has length* 2^{-2^n}*; and let* $\tilde{S}$ *denote the cumulative distribution function of the measure still assigning weight* 2^{-n} *to each* $\tilde{I}_{n,k}$*. Then* $[0;1]^2 \ni (x,y) \mapsto \int_x^y f(t)\, d\tilde{S}(t)$ *is polynomial-time computable for every polynomial-time computable* $f : [0;1] \to \mathbb{R}$.

Note that $\tilde{S}$ has an exponential, but no polynomial, modulus of continuity; and $\tilde{\mathcal{C}}$ has Hausdorff dimension 0: in fact it has logarithmic *metric entropy* in the following sense:

Definition 3. *Let* (X,d) *denote a metric space and* $\mathrm{ball}(x,\varepsilon) := \{x' : d(x,x') < \varepsilon\}$.

a) A metric entropy $\eta : \mathbb{N} \to \mathbb{N}$ *of* X *satisfies that, to every* $n \in \mathbb{N}$*, there exist* $x_1, \ldots, x_{2^{\eta(n)}} \in X$ *with* $X \subseteq \bigcup_{1\leq j\leq 2^{\eta(n)}} \overline{\mathrm{ball}}\big(x_j, 2^{-n}\big)$*; cmp.* [Weih03, Definition 6.2].

b) The support *of a Borel measure* μ *is* $\mathrm{supp}(\mu) := \big\{x \in X \;\big|\; \forall \varepsilon > 0 : \mu\big(\mathrm{ball}(x,\varepsilon)\big) > 0\big\}$.

Using a result on the complexity of measures from [FGH13] we can generalize Proposition 2c):

Theorem 4. *Let* μ *be a measure over* $[0;1]$ *such that the its support has logarithmic metric entropy. Then in-definite* μ*-integration is polynomial time computable relative to some oracle.*

In particular, the support of such a measure has Hausdorff dimension 0 (and thus measure 0). Examples of such measures include the previously defined measure supported on $\tilde{\mathcal{C}}$, or the finite discrete measures, whose support can be covered by a bounded number of such balls, independently from n.

Note that the converse of Theorem 4 is not true, since a discrete measure with faster speed of convergence (for example $\nu_1(f) = \sum_{j\in\mathbb{N}} f(k \cdot 2^{-\lceil \log_2 k\rceil}) \cdot 2^{-k}$) can be polynomial time computable even if its support is dense in $[0;1]$.

In most examples, measures which do not make integration relatively polynomial time computable make it **#P**-hard. However, we do not have a relevant sufficient condition for **#P**-hard integration in general.

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Operators for BSS RAM's

Christine Gaßner and Pedro Francisco Valencia Vizcaíno

Ernst-Moritz-Arndt-Universität Greifswald, Germany
E-mails: gassnerc@uni-greifswald.de, p.f.valencia.vizcaino@uni-greifswald.de

In this exposition we present some recent advances in the investigation of abstract computation over mathematical structures on the basis of the BSS RAM model. This model combines the advantages of the uniform treatment of problem instances with the possibilities to describe practical relevant algorithms on a high level of abstraction according with high-level programming languages. The considered model is based on the design of the well-known random access machines and is a generalization of both the uniform BSS model of computation and the Turing machine. Since these models play an important role for analysing the complexity of algorithms, we want to study the typical features of our model for some structures, as well as to compare it, for several structures, with other models of computation, and to attempt a classification of undecidable problems. Here we would like to follow up on our discussion (in [Gaßner '15]) on classes of problems resulting from the transfer of the classical arithmetical hierarchy over the non-negative integers to other algebraic structures. In particular, we will study operators introduced by Stephen Cole Kleene, by Yiannis Nicholas Moschovakis, their counterparts within the framework of our BSS RAM's, similar new operators, and the relationship between nondeterministic BSS RAM's and Moschovakis' operator.

We consider structures $\mathcal{A} = (U; (d_i)_{i\in I}; f_1, f_2, \ldots; r_1, r_2, \ldots)$ such that there are at least two constants $d_0, d_1 \in U$ (with $0, 1 \in I$). Any *BSS RAM over* $\mathcal{A}$ or $\mathcal{A}$*-machine* can process inputs $(x_1, \ldots, x_n) \in U^\infty =_{\rm df} \bigcup_{i=1}^{\infty} U^i$ of any length and we assume that each function f_i can be executed by any machine over $\mathcal{A}$ within one step and that each relation r_i can be thus evaluated. At the beginning, besides the input $(x_1, \ldots, x_n)$, the first index register also obtains the length n by an input procedure, and moreover, the *nondeterministic* machines are able to guess an arbitrary finite number m of arbitrary elements $y_1, \ldots, y_m \in U$ right after the input procedure by executing a guessing procedure. For the classes $\mathsf{M}_\mathcal{A}$ and $\mathsf{M}_\mathcal{A}^{\rm N}$ of all deterministic and, respectively, nondeterministic machines over $\mathcal{A}$, we consider the halting problems $\mathbb{H}_\mathcal{A}$ and $\mathbb{H}_\mathcal{A}^{\rm N}$ where we write $(\vec{x}\,.\,c_\mathcal{M})$ instead of $(x_1, \ldots, x_n, s_1, \ldots, s_k)$ if $\vec{x} = (x_1, \ldots, x_n)$ and $c_\mathcal{M} = {\rm code}(\mathcal{M}) = (s_1, \ldots, s_k) \in U^\infty$ is the code of $\mathcal{M}$ given by a sequence of the codes of the single symbols of the program of $\mathcal{M}$ in analogy with [Blum et al. '89] (where each constant d_i is encoded by itself and each of the other single symbols is unambiguously encoded by tuples in $\{d_0, d_1\}^\infty$). $\mathcal{M}(\vec{x})\downarrow$ means that $\mathcal{M}$ halts on $\vec{x}$ (for some guesses in the nondeterministic setting).

$$\mathbb{H}_{\mathcal{A}}^{[\mathrm{N}]} = \{(\vec{x}\,.\,c_{\mathcal{M}}) \mid \vec{x} \in U^\infty \;\&\; \mathcal{M} \in \mathsf{M}_{\mathcal{A}}^{[\mathrm{N}]} \;\&\; \mathcal{M}(\vec{x})\downarrow\}$$

On the one hand, there are close relations between these halting problems. $\mathbb{H}_{\mathcal{A}}^{\mathrm{N}}$ can, for instance, be reduced to $\mathbb{H}_{\mathcal{A}}$ by a BSS RAM using $\mathbb{H}_{\mathcal{A}}$ as oracle if the problems in the corresponding class $\mathrm{NP}_{\mathcal{A}}$ are decidable over $\mathcal{A}$. And moreover, it is possible to many-one-reduce the following problems to each other by machines over $\mathcal{A}$: $\mathbb{H}_{\mathcal{A}}^{\mathrm{N}} \equiv_{\mathcal{A}} \mathbb{H}_{\mathcal{A}}^{\mathrm{PROJ}} \equiv_{\mathcal{A}} \mathbb{H}_{\mathcal{A}}^{\mathrm{EXI}}$ (where $\bar{n}$ stands for the n-tuple $(d_0, \ldots, d_0, d_1)$).

$$\mathbb{H}_{\mathcal{A}}^{\mathrm{EXI}} = \{(\vec{x}\,.\,c_{\mathcal{M}}) \mid \vec{x} \in U^\infty \,\&\, \mathcal{M} \in \mathsf{M}_{\mathcal{A}} \,\&\, (\exists \vec{y} \in U^\infty)(\mathcal{M}(\vec{x}\,.\,\vec{y})\downarrow)\}.$$

$$\mathbb{H}_{\mathcal{A}}^{\mathrm{PROJ}} = \bigcup_{n=1}^{\infty}\{(\vec{x}\,.\,c_{\mathcal{M}}) \mid \vec{x} \in U^n \,\&\, \mathcal{M} \in \mathsf{M}_{\mathcal{A}} \,\&\, (\exists \vec{y} \in U^\infty)(\mathcal{M}(\bar{n}\,.\,\vec{x}\,.\,\vec{y})\downarrow)\}.$$

On the other hand we can extend the BSS RAM's by an operator such that a new type of instructions allows to compute *multiple-valued functions* from U^∞ to the power set of U^∞. In analogy with the operator ν introduced by Moschovakis in [Moschovakis '69], we will permit the application of an operator ν and, for partially ordered structures, further operators which we call the *Moschovakis operators*. In contrast to Kleene's definition of a μ-operator where an integer is searched, we take into consideration that most tuples of individuals cannot be represented by a single individual. For the structure $\mathcal{A}$ and any function $f : U^\infty \to U$ computable by a usual BSS RAM over $\mathcal{A}$, we allow that *ν-oracle BSS RAM's* can — in addition to the instruction types of the usual BSS RAM's — execute *ν-instructions* of the form

$$\ell: \; Z_j := \nu[f](I_1, Z_1, \ldots, Z_{I_1}).$$

In this way, the register Z_j obtains the value y_1 if the content of I_1 is n, the contents of $Z_1, \ldots, Z_n$ are $z_1, \ldots, z_n$, and $y_1 \in U$ is some value such that there are an m and $y_2, \ldots, y_m \in U$ satisfying $f(z_1, \ldots, z_n, y_1, \ldots, y_m) = d_0$, and otherwise Z_j does not obtain a value and the machine again goes to the label ℓ. If, with regard to these machines, we say that a problem $P \subseteq U^\infty$ is *ν-semi-decidable over $\mathcal{A}$* by a ν-oracle BSS RAM $\mathcal{M}$ over $\mathcal{A}$ when $\mathcal{M}$ computes a multiple-valued function f such that $d_0 \in f(\vec{x})$ iff $\vec{x} \in P$, then we obtain the following relations.

1. For any ν-oracle BSS RAM ν-semi-deciding a problem over $\mathcal{A}$, there is a nondeterministic $\mathcal{A}$-machine that recognizes the same problem.

2. If $\mathcal{A}$ is a structure with at least two constants, then, for any nondeterministic $\mathcal{A}$-machine recognizing a problem, there is a ν-oracle BSS RAM that ν-semi-decides the same problem over $\mathcal{A}$.

In our presentation, we introduce characterizations of some Moschovakis operators and consider some classes of hierarchies defined in analogy with the arithmetical hierarchy.

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Computation of the asymptotic behavior of dynamical systems

Daniel S. Graça* Ning Zhong†

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A common problem in dynamical systems theory is to characterize the regions of space to which the trajectories of an ordinary differential equation $x' = f(x)$ may converge. This has applications in many fields, such as in verification problems, where one is interested in knowing whether a system starting in some initial state may eventually reach an "unsafe" state. The latter problem leads naturally to the following question: given the description of some dynamical system as input, is it possible to compute its "limit set", i.e., the set of points to which the trajectories will eventually converge? This "limit set" corresponds to the notion of non-wandering set, $NW(f)$. Therefore, the question can be restated as follows: given a description of some dynamical system, is it possible to compute its non-wandering set?

The uniform computation of $NW(f)$ is in general not possible, due to continuity problems. In fact, except for some very particular classes of dynamical systems, no general theory exists to this day about the characterization of non-wandering sets for dynamical systems defined on spaces of dimension greater than two; although there are some emerging theories - still in their infancy - for the three-dimensional case. On the other hand, there is a much celebrated characterization for non-wandering sets in the two-dimensional case established by Peixoto in 1962 [Pei62], in which Peixoto showed that (i) the set of structurally stable systems defined on a compact $K \subseteq \mathbb{R}^2$ is an open and dense subset of $C^1(K)$; and (ii) the non-wandering set of a structurally stable system defined on K consists only of hyperbolic equilibrium points and of hyperbolic periodic orbits.

From the computational perspective, Peixoto's theorem also provides valuable information. Firstly, since the structurally stable systems defined

*CEDMES/FCT, Universidade do Algarve, C. Gambelas, 8005-139 Faro, Portugal & SQIG/Instituto de Telecomunicações, Lisbon, Portugal.

†DMS, University of Cincinnati, Cincinnati, OH 45221-0025, U.S.A.

on K form an open and dense subset of $C^1(K)$ (all C^1 systems defined on K), it makes sense to pursue uniform computation of $NW(f)$ of structurally stable systems; thus avoiding continuity problems triggered by the unstable systems. Secondly, knowing exact constituents of $NW(f)$ (only hyperbolic equilibria and periodic orbits) provides us a number of existing tools potentially useful in the construction of an algorithm to compute $NW(f)$. The talk will be devoted to this case, where several ideas and techniques will be introduced, with the aim of showing the following result: the non-wandering set $NW(f)$ is (uniformly) computable for structurally stable dynamical systems $x' = f(x)$ defined on a compact $K \subseteq \mathbb{R}^2$.

To (uniformly) compute $NW(f)$ it is enough to show that the set of zeros (equilibria) of f is computable as well as the set of all periodic orbits of the dynamical system $x' = f(x)$. In general, it is not possible to uniformly compute the zeros of a function. However, since all equilibria of a planar structurally stable system are hyperbolic, it follows that if x_0 is a zero of f, then the jacobean $Df(x_0)$ is invertible. This crucial fact enables us to show that the equilibria of f are computable. In fact, the following more general result is proved: the set $\{x \mid f(x) = 0, \; Df(x) \neq 0\}$ is computable.

The later result is proved by constructing an algorithm that discretizes the state space into small "pieces" and then computes over- and under- approximations of $f(s)$ for each "piece" s. Using the inverse function theorem (this is where the fact of $Df(x_0)$ being invertible is used) we can show that for a "small enough" state space discretization, the algorithm always halts and returns the zeros of f.

To compute periodic orbits, an intuitive idea is to sample the trajectories of the system $x' = f(x)$ periodically with some time interval $\Delta > 0$. This idea indeed works if the period of a periodic orbit happens to be a multiple or a divisor of Δ. However, since we may not have any information on the periods prior to computation, we use instead a more elaborated technique from [PBV95]. The advantage of this technique is that it allows one to (over-)approximate the dynamics of $x' = f(x)$ by a graph. Yet, since the technique only generates approximations, it may produce non-existing "fake" periodic orbits or it generates a "real" periodic orbit but fails to provide information on accuracy of approximation. We solve these problems by employing various techniques such as coloring the algorithms for detecting the "inside" and "outside" of a periodic orbit and then using the information obtained to compute the accuracy of the generated approximate orbits. By utilizing the hypothesis that the system is structurally stable alongside with several other techniques, we are able to show that the occurrence of "fake" periodic orbits can be avoided with a good enough over-approximation of the dynamics of $x' = f(x)$.

The talk is based on work in progress.

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The Baire property holds in the projective hierarchy almost uniformly (Abstract)

Vassilios Gregoriades
TU Darmstadt
gregoriades@mathematik.tu-darmstadt.de

We employ the standard *projective hierarchy* $\utilde{\mathbf{\Sigma}}^1_n$, $n \in \mathbb{N}$, of sets in Polish spaces, see for example [2] for the definition. The class of *analytic sets* is $\utilde{\mathbf{\Sigma}}^1_1$. It is a well-known result of Lusin-Sierpinsky that analytic sets have the Baire property, i.e., for all Polish spaces $\mathcal{X}$ and all analytic $P \subseteq \mathcal{X}$ there exists some open $U \subseteq \mathcal{X}$ such that the symmetric difference $P \triangle U := (P \setminus U) \cup (U \setminus P)$ is a meager set. Moreover it is also well-known that every set in $\cup_{n\in\mathbb{N}} \utilde{\mathbf{\Sigma}}^1_n$ has the Baire property under the axiom of Projective Determinacy (i.e., in every Gale-Stewart game, whose payoff set belongs to $\cup_{n\in\mathbb{N}} \utilde{\mathbf{\Sigma}}^1_n$, one of the two players has a winning strategy). This result follows from work of Mazur, Banach, Oxtoby, Mycielski and Steinhaus; the reader may refer to [2, Ch. 6] for more information.

We are concerned with the problem of witnessing the Baire property of the projective hierarchy (using determinacy hypothesis where needed) in a uniform way. A standard example in the project of establishing structural properties in the projective hierarchy in a uniform way is the *Suslin-Kleene Theorem.* The classical *Suslin Theorem* states that every bi-analytic subset of a Polish space is Borel, while the Suslin-Kleene Theorem gives a recursive (and thus continuous) function $u : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$, such that whenever α and β are codes of complementary analytic sets, say A and $\mathcal{X} \setminus A$, then $u(\alpha, \beta)$ is a Borel code of A, cf. [2, 7B.4].

The notion of a code is given through universal (or parametrization) sets. We proceed to the necessary definitions.

Suppose that $\mathcal{X}$ and $\mathcal{Z}$ are Polish spaces and that Γ is a class of sets. We denote by $\Gamma \upharpoonright \mathcal{X}$ the family of all subsets of $\mathcal{X}$ which are in Γ.

A set $G \subseteq \mathcal{Z} \times \mathcal{X}$ *parametrizes* $\Gamma \upharpoonright \mathcal{X}$ if for all $P \subseteq \mathcal{X}$ we have that

$$P \in \Gamma \iff \text{exists } z \in \mathcal{Z} \text{ such that } P = \{x \mid (z, x) \in G\}.$$

Any z as above is called a Γ-*code* of P. We will denote the z-section $\{x \mid (z, x) \in G\}$ of G with $G(z)$.

The set G is *universal* for $\Gamma \upharpoonright \mathcal{X}$ if G is in Γ and parametrizes $\Gamma \upharpoonright \mathcal{X}$. A *universal system* for Γ is an assignment $\mathcal{Y} \mapsto G^{\mathcal{Y}} \subseteq \mathbb{N}^{\mathbb{N}} \times \mathcal{Y}$ such that the set $G^{\mathcal{Y}}$ is universal for $\Gamma \upharpoonright \mathcal{Y}$ for all Polish spaces $\mathcal{Y}$.

The classes of open sets and $\utilde{\mathbf{\Sigma}}^1_n$, $n \in \mathbb{N}$, admit universal systems in a natural way. We fix such systems, so that when we refer e.g. to a $\utilde{\mathbf{\Sigma}}^1_1$-code this will be with respect to the good universal system for $\utilde{\mathbf{\Sigma}}^1_1$. (The definition of a Borel code is a bit more elaborate cf. [2, 3H, 7B].)

We say that a class of sets Γ has the *Baire property* if for all Polish spaces $\mathcal{X}$ every set $P \in \Gamma \upharpoonright \mathcal{X}$ has the Baire property. The following shows that one can witness the Baire property in reasonable classes of sets almost uniformly.

Lemma 1. *Assume that* Γ *admits a universal system* $(G^{\mathcal{Y}}_\Gamma)_{\mathcal{Y}}$ *and that it has the Baire property. Then for every Polish space* $\mathcal{X}$ *there exists a continuous function* $u^{\mathcal{X}}_\Gamma : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ *such that for* almost all $\alpha \in \mathbb{N}^{\mathbb{N}}$ *the set* $G^{\mathcal{X}}_\Gamma(\alpha) \triangle G^{\mathcal{X}}_{\utilde{\mathbf{\Sigma}}^0_1}(u^{\mathcal{X}}_\Gamma(\alpha))$ *is meager.*

This has the following consequence.

Proposition 2 (Axiom of Projective Determinacy for $n > 1$)**.** *For every Polish space $\mathcal{X}$ and every $n \in \mathbb{N}$ there exists continuous function $u_n^{\mathcal{X}} : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that for* almost all $\alpha \in \mathbb{N}^{\mathbb{N}}$ *the set $G^{\mathcal{X}}_{\utilde{\Sigma}^1_n}(\alpha) \triangle G^{\mathcal{X}}_{\utilde{\Sigma}^0_1}(u_n^{\mathcal{X}}(\alpha))$ is meager.*

Remark 3. Since the uniformity function $u_\Gamma^{\mathcal{X}}$ in Lemma 1 is continuous, it is in particular A-recursive for some $A \subseteq \omega$. It actually follows from the proof of this lemma that the latter set A is derived from any open set which witnesses the Baire property for the universal set $G^{\mathcal{Y}}_\Gamma$. The complexity of such an A is typically high: even in the case of $\Gamma = \utilde{\Sigma}^1_1$ it seems that any such A must be at least as complex as a (true) Π^1_1 subset of ω. It would be interesting to investigate if the preceding A can be chosen to be recursive in a Π^1_1 subset of ω, and if this bound is tight, i.e., one cannot choose it to be Δ^1_1.

The next question is whether one can improve upon the "for almost all" part of the preceding results and turn it into "for all". It turns out that such an improvement is impossible in any level of the projective hierarchy even if the uniformity function is allowed to be measurable with respect to that level and not necessarily continuous.

Theorem 4. *For every $n \geq 1$ there is no $\utilde{\Delta}^1_n$-measurable function $u : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that the set $G^{\mathbb{N}^{\mathbb{N}}}_{\utilde{\Sigma}^1_n}(\alpha) \triangle G^{\mathbb{N}^{\mathbb{N}}}_{\utilde{\Sigma}^0_1}(u(\alpha))$ is meager* for all $\alpha \in \mathbb{N}^{\mathbb{N}}$.

Topic for future research. The preceding results essentially answer the question of witnessing the Baire property in a uniform way in the projective classes. The next question is whether we can replace the Baire property with other structural properties. The case of the separation property of analytic sets has been answered by the Suslin-Kleene Theorem, but it is nevertheless interesting to pose the same question for special separation theorems.

Two such characteristic results are due to Dyck and Preiss, cf. [1, 28.12, 28.14]. Here we announce the constructive version of Dyck's result.

For all $n \in \mathbb{N}$ we define the subsets of $2^{\mathbb{N}}$, $U_n = \{x \in 2^{\mathbb{N}} \mid x(n) = 1\}$. Given a family $\mathcal{F}$ of sets we denote by $(\sigma, \delta)(\mathcal{F})$ the least family of sets which contains $\mathcal{F}$ and is closed under countable intersections as well as countable unions. For $x, y \in 2^{\mathbb{N}}$ we write $x \subseteq y$ if the set $\{n \in \mathbb{N} \mid x(n) = 1\}$ is contained in $\{n \in \mathbb{N} \mid y(n) = 1\}$.

A set $A \subseteq 2^{\mathbb{N}}$ is *positive* if it belongs to $(\sigma, \delta)(\{U_n \mid n \in \mathbb{N}\})$, and is *monotone* if for all $x, y \in 2^{\mathbb{N}}$ with $x \in A$ and $x \subseteq y$ we have $y \in A$. It is easy to verify that every positive set is monotone. The converse is also true for sets which differ from $\emptyset$ and $2^{\mathbb{N}}$. This is a corollary to the Dyck Separation Theorem cf. [1, 28.12].

Theorem 5 (The effective Dyck Separation Theorem)**.** *There exists a recursive (and thus continuous) function $u : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that whenever α, β are $\utilde{\Sigma}^1_1$ codes of non-empty disjoint (analytic) sets A, B respectively, then $u(\alpha, \beta)$ is a Borel code of a positive set C such that $A \subseteq C$ and $C \cap B = \emptyset$.*

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On the Image and Length of Polynomial-Time Computable Curves*

Tomohiro Katayama   Akitoshi Kawamura

A curve in the plane $\mathbb{R}^2$ is said to be (polynomial-time) computable if it has a (polynomial-time) computable parametrization. We investigate the complexity and computability of the image and the length of polynomial-time computable simple curves. We relate the complexity of the image of such curves with the complexity classes NP and UP. We then show that the intersection of the image of two polynomial-time computable curves is not necessarily computable. We also show that, while the length of a polynomial-time computable curve is known to be non-computable in general [8, 4, 14], it has complexity equivalent to $\#\mathrm{P}_1$ (the tally version of the counting class #P) if we further assume that the derivative of the curve is polynomial-time computable. Details follow.

We study the complexity of the image of polynomial-time computable curves under the following two notions of computing real sets. We say that oracles ϕ, ψ represent a point $x = (x_1, x_2) \in \mathbb{R}^2$ if $|\phi(1^n) - x_1| < 2^{-n}$ and $|\psi(1^n) - x_2| < 2^{-n}$ for all $n \in \mathbb{N}$. We write $B(x, r)$ for the ball $\{\, y \in \mathbb{R}^2 \mid |x - y| \leq r \,\}$.

Definition 1 ([2])**.** A set $S \subseteq \mathbb{R}^2$ is *strongly recognizable* if there is an oracle Turing machine $M^{\phi,\psi}$ that satisfies

$$M^{\phi,\psi}(n) = \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{if } B(x, 2^{-n}) \cap S = \emptyset, \\ 0 \text{ or } 1 & \text{otherwise} \end{cases} \tag{1}$$

for any oracles ϕ, ψ that represent a point $x \in \mathbb{R}^2$.

Definition 2 ([1][1])**.** A set $S \subseteq \mathbb{R}^2$ is *locally computable* if there is an oracle Turing machine $M^{\phi,\psi}$ that satisfies

$$M^{\phi,\psi}(n) = \begin{cases} 1 & \text{if } B(x, 2^{-n}) \cap S \neq \emptyset, \\ 0 & \text{if } B(x, 2 \cdot 2^{-n}) \cap S = \emptyset, \\ 0 \text{ or } 1 & \text{otherwise} \end{cases} \tag{2}$$

for any oracles ϕ, ψ that represent a point $x \in \mathbb{R}^2$.

We say a set S is *strongly* P*-recognizable* (and *locally* P*-computable*, respectively) if the machine in Definition 1 (and in Definition 2, respectively) runs in polynomial time for all inputs. We can see from the definitions that local computability implies strong recognizability, and that local P-computability implies strong P-recognizability. In fact, local computability and strong recognizability are equivalent. However, strong P-recognizability and local P-computability are equivalent if and only if P = NP [1].

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[1] The original definition in [1] is slightly different in that the number x, which we represent as oracles in Definition 2, is restricted to rational numbers and given to the machine as strings. It can be shown that the two definitions are equivalent.

Theorem 1. *In the following, (a) $\Leftrightarrow$ (b) and (a) $\Rightarrow$ (c) $\Rightarrow$ (d).*

(a) $\mathrm{P} = \mathrm{NP}$.
(b) *For any polynomial-time computable curve* $f\colon [0,1] \to \mathbb{R}^2$, *its image* $f([0,1])$ *is locally* P*-computable.*
(c) *For any polynomial-time computable curve* $f\colon [0,1] \to \mathbb{R}^2$, *its image* $f([0,1])$ *is strongly* P*-recognizable.*
(d) $\mathrm{P} = \mathrm{UP}$.

A similar investigation was done by Braverman [1] for the graph of polynomial-time computable functions. We obtain Theorem 1 by using this result and encoding a UP set similarly to [2] and [12].

For the intersection, we prove the following.

Theorem 2. *There exist two polynomial-time computable curves such that the intersection of their images is not strongly recognizable.*

Since local computability and strong recognizability are equivalent [1], Theorem 2 holds if we replace the phrase "strongly recognizable" to "locally computable". It is known that the intersection of two strongly recognizable sets is not computable in general [13]. Theorem 2 indicates that this is still true if we restrict the sets to polynomial-time computable curves. We use the polynomial-time computable function in [7] which has an uncountable number of roots, and prove that the intersection of its graph and the line segment $[0,1] \times \{0\}$ is not strongly recognizable.

Finally, for the length, we prove the following.

Theorem 3. *The following are equivalent.*

(a) $\mathrm{FP}_1 = \#\mathrm{P}_1$.
(b) *Any polynomial-time computable curve which has a polynomial-time computable derivative has polynomial-time computable length.*

This means that computing the length of a polynomial-time computable curve with polynomial-time computable derivative is as hard as computing the integral of a polynomial-time computable function [3]. We note that previously, the length of computable curves has mainly been studied in terms of its computability [8, 4, 14].

Our approach overall is in line with the method initiated by Ko and Friedman [11], and developed in subsequent papers [5, 6, 8, 9, 2, 10, 1].

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Towards Computational Complexity Theory on Advanced Function Spaces in Analysis*

Akitoshi Kawamura[1], Florian Steinberg[1,2], and Martin Ziegler[2]

[1] The University of Tokyo [2] TU Darmstadt

Abstract. We present a concise 2nd-order representation (i.e. over finite string functions rather than infinite binary strings) and 2nd-order parametrization (i.e. over integer functions rather than over integers) of the Sobolev spaces $W_p^k[0;1]$; and we give various evidence that this is the 'right' choice for complexity-theoretic studies, particularly of partial differential equations.

TTE compares and studies transformation properties of representations for a given space X. In particular several natural but different representations of spaces of continuous functions on Euclidean domains have been established as computably equivalent. Partial differential equations, however, exhibit counter-intuitive computability properties when considered on such classical function spaces rather than than the advanced ones suggested by functional analysis: L^p and more generally Sobolev spaces W_p^k [WeZh02]. The qualitative computability theory of such spaces being well established [SZZ15], complexity-theoretic refinements bridge the gap to numerical practice [KORZ14]. In fact many classical notions and complexity classes translate naturally to the continuous case [Ko91, Weih03]; with two amendments:

• While computations on separable compact metric spaces admit complexity bounds depending on $n \in \mathbb{N}$ only, σ-compact spaces require (appropriate!) additional integer parameters $k \in \mathbb{N}$ [KMRZ15] and function spaces even 2nd-order parameters $K \in \mathbb{N}^{\mathbb{N}}$ [KaCo10, KaPa14].

• For representing function spaces, sequential access via infinite binary strings be replaced by oracle access, i.e. 2nd-order representations over finite string functions [KaCo10, KaPa14].

Whether and how processing an oracle with 'long' answers should be alotted more time is an ongoing debate. We avoid it by restricting to binary oracles $\mathcal{O} : \{0,1\}^* \to \{0,1\}$ and binary 2nd-order representations $\boldsymbol{\delta} :\subseteq \{0,1\}^* \to \{0,1\}$. Real numbers are implicitly considered equipped with the standard representation [Weih03, Examples 4.2.2+4.2.3].

Definition 1. *a) A mapping $\mu : \mathbb{N} \to \mathbb{N}$ is a* modulus of continuity *to a function $f : X \to \mathbb{R}$ if, for every $n \in \mathbb{N}$ and $x, x' \in X$, $d(x,x') \le 2^{-\mu(n)}$ implies $\big|f(x) - f(x')\big| \le 2^{-n}$.*

b) The outer metric entropy *of a totally bounded metric space (X,d) is the mapping $\lceil X \rceil : \mathbb{N} \to \mathbb{N}$ s.t. X can be covered by $2^{\lceil X \rceil(n)}$ open balls of radius 2^{-n}, but not by $2^{\lceil X \rceil(n)-1}$.*

c) The inner metric entropy *of totally bounded (X,d) is the mapping $\lfloor X \rfloor : \mathbb{N} \to \mathbb{N}$ such that, for every $n \in \mathbb{N}$, X admits $2^{\lfloor X \rfloor(n)}$, but not $2^{\lfloor X \rfloor+1}$, points of pairwise distance $\ge 2^{-n}$.*

*d) Fix $\Omega \subseteq \mathbb{R}^d$. A mapping $\nu : \mathbb{N} \to \mathbb{N}$ is an L^p–*modulus *to $f \in L^p(\Omega)$ if $\|\boldsymbol{h}\|_\infty \le 2^{-\nu(n)}$ implies $2^{-pn} \ge \int_\Omega \big|f(\boldsymbol{x}+\boldsymbol{h}) - f(\boldsymbol{x})\big|\, d\boldsymbol{x}$ with the convention $f\big|_{\mathbb{R}^d \setminus \Omega} :\equiv 0$.*

Recall that totally bounded is equivalent to precompact. One can prove $\lfloor X \rfloor(n) \le \lceil X \rceil(n) \le \lfloor X \rfloor(n+1)$; compare the slightly less tight bounds in [Weih03, §6]. In particular we may in the sequel ambiguously refer to 'the' metric entropy as being polynomial or exponential.
We employ the BachmannLandau notation $f \in \Theta(g) \;\Leftrightarrow\; f \in \mathcal{O}(g) \;\wedge\; g \in \mathcal{O}(f)$.

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Proposition 2. *a) The hypercubes* $[0;2^k]^d$ *have asymptotic metric entropy* $\Theta\big(d\cdot(n+k)\big)$.

b) The compact set of non-expansive $f:[0;1]\to[0;1]$, *equipped with the supremum norm, has exponential metric entropy:* $\lceil \mathrm{Lip}_1([0;1],[0;1])\rceil(n)=2^{\Theta(n)}$. *For* $\mu:\mathbb{N}_0\to\mathbb{N}$, *the set*

$$C_\mu[0;1] \;:=\; \big\{f\in C[0;1] \;\big|\; \|f\|_\infty\leq 2^{\mu(0)} \;\wedge\; \mu \text{ modulus of continuity to } f\big\}$$

has asymptotic metric entropy $\lceil C_\mu[0;1]\rceil(n)=2^{\Theta(\mu(n\pm 1))}$.
A subset of $C[0;1]$ *is relatively compact iff it is contained in some* $C_\mu[0;1]$.

c) Similarly for $\nu:\mathbb{N}_0\to\mathbb{N}$, *with respect to the* L^p*–norm the set*

$$L^p_\nu[0;1] \;:=\; \big\{f\in L^p[0;1] \;\big|\; \|f\|_\infty\leq 2^{\nu(0)} \;\wedge\; \nu \text{ an } L^p\text{–modulus to } f\big\}$$

has asymptotic metric entropy $\lceil L^p_\mu[0;1]\rceil(n)\cong 2^{\Theta(\nu(n\pm 1))}$.
And a subset of $L^p[0;1]$ *is relatively compact iff it is contained in some* $L^p_\nu[0;1]$.

d) Every precompact metric space (X,d) *admits an equivalent metric* d' *such that* (X,d') *has polynomial metric entropy. For a compact subset* X *of a vector space equipped with two equivalent* norms $\|\cdot\|$ *and* $\|\cdot\|'$, *however, their induced metric entropies can differ by at most a constant shift and offset:*

$$\exists c\in\mathbb{N}\;\forall n\geq c:\quad \lceil X'\rceil(n-c)-c\;\leq\;\lceil X\rceil(n)\;\leq\;\lceil X'\rceil(n+c)+c\;.$$

Item b) constitutes a quantitative refinement of the classical Arzelá–Ascoli Theorem; and Item c) of the Fréchet–Kolmogorov Theorem. We also exhibit the following connections between metric entropy and computational complexity.

Theorem 3. *For* (X,d) *a precompact metric space, the following are equivalent:*

i) X *has at most polynomial metric entropy, that is,* $\lceil X\rceil(n)\leq p(n)$ *for some* $p\in\mathbb{N}[N]$.

ii) There exists a (1st-order) representation $\delta :\subseteq \{\mathtt{0},\mathtt{1}\}^\omega \twoheadrightarrow X$ *rendering the following multi-valued/partial parameterized equality test computable in time polynomial in* n*:*

$$X\times X\times\mathbb{N}\;\ni\;(x,y,n)\;\mapsto\;\mathtt{true}\text{ for } x=y,\quad (x,y,n)\mapsto\;\mathtt{false}\text{ for } d(x,y)\geq 2^{-n}.\quad (1)$$

iii) There exists a (1st-order) representation $\delta :\subseteq \{\mathtt{0},\mathtt{1}\}^\omega \twoheadrightarrow X$ *rendering the metric* $d: X\times X\to[0;\infty)$ *polynomial-time computable relative to some oracle.*

iv) There exists a (1st-order) representation δ *of* X *and polynomial* $p\in\mathbb{N}[N]$ *with the following universal property: If* $f:X\to[0;1]$ *has modulus of continuity* $\mu:\mathbb{N}\to\mathbb{N}$, *then* f *is computable in time* $p\big(\mu(n)+n\big)$ *relative to some appropriate oracle, which in turn implies* f *to have modulus of continuity* $p\big(p\big(\mu(p(n))+n\big)\big)$.

v) There exists a binary 2nd-order representation $\boldsymbol{\delta}$ *of* X *rendering the metric/test (1) computable in (1st-order) polynomial time.*

Item (iv) extends [Ko91, Theorem 2.19], [Weih00, Exercise 7.1.7], and [PaZi13, Lemma 6.3]. Surprisingly, exponential metric entropy corresponds to polynomial space complexity with 2nd-order representations:

Theorem 4. *For* (X,d) *a precompact metric space, the following are equivalent:*

i) X *has at most exponential metric entropy, that is,* $\lceil X\rceil(n)\leq 2^{p(n)}$ *for some* $p\in\mathbb{N}[N]$.

ii) There exists a binary 2nd-order representation $\boldsymbol{\delta}$ *of* X *rendering the partial parameterized equality test (1) computable within* space *polynomial in* n.

iii) *There exists a binary 2nd-order representation* $\boldsymbol{\delta}$ *of* X *rendering the metric computable within (1st-order) polynomial space by an oracle machine with constant-depth stack of write-only query tapes exempt from space bounds; cmp.* [KaOt14, p.437] *and* [Blas88].
iv) *There is a binary 2nd-order representation* $\boldsymbol{\delta}$ *of* X *and a* $p \in \mathbb{N}[N]$ *with this universal property: If* $f : X \to [0;1]$ *has modulus* μ*, then* f *is computable in space* $p\big(\mu(n)+n\big)$ *by a machine as in (iii) relative to some oracle, then* f *has modulus* $p\big(p(\mu(p(n))+n)\big)$.
v) *There is a 1st-order representation* δ *of* X *rendering the metric/test (1) relatively computable in (1st-order) exponential time.*

Since the minimal modulus of continuity determines the relative computational complexity of continuous functions, it has been employed as a 2nd-order parameter for complexity investigations of operators on functions [KaCo10, Lemma 4.9]. In view of Item (iv) and the analogy between Proposition 2b) and c), it seems natural to use the L^p–modulus as 2nd-order parameter for a complexity theory of L^p and, more generally, Sobolev spaces $W_p^k[0;1]$.

Definition 5. *For* $k \in \mathbb{N}$ *let* $\boldsymbol{\delta}_p^k[0;1]$ *denote the following 2nd-order representation of* $W_p^k[0;1]$*: a name* $\psi : \{0,1\}^* \to \{0,1\}^*$ *of* f *encodes, for each* $n, m \in \mathbb{N}$ *given in unary and* $a \in \mathbb{Z}$ *in binary, some dyadic* 2^{-n}*–approximation to* $\int_{[a/2^n-2^{-m};a/2^n+2^{-m}]\cap[0;1]} f(t)\,dt$ *and has length* $|\psi|$ *an* L^p*–modulus of the* k*-th weak derivative of* f *plus an upper bound on* $\|f\|_p$ *in binary.*

Theorem 6. *Both weak differentiation* $W_1^{k+1}[0;1] \ni f \mapsto f' \in W_1^k[0;1]$ *and the Sobolev embeddings* $W_1^{k+1}[0;1] \hookrightarrow C^k[0;1]$ *are 2nd-order polynomial-time* $(\boldsymbol{\delta}_1^{k+1}, \boldsymbol{\delta}_1^k)$*–computable.*

Recall $\|f\|_{W_p^k} = \|f\|_p + \|f^{(k)}\|_p$. According to Proposition 2c) and Theorem 3iii), the L^p–norms and function differences cannot simultaneously be polynomial-time computable. Future work will extend the above considerations from $W_1^k[0;1]$ to $W_p^k(\Omega)$ for $p > 1$ and convex $\Omega \subseteq [0;1]^d$.

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Effective zero-dimensionality and retracts

Robert Kenny

In our recent work, some effective versions of conditions for zero-dimensionality of a computable metric space X have been considered in computable analysis via representations; each of the (classically equivalent) properties of having vanishing small inductive dimension, large inductive dimension or covering dimension, or having a countable basis of clopen sets, can be interpreted as a multi-valued operation, and the computability of these operations (as well as certain natural variants) has been shown to be mutually equivalent [1, Prop 5.1]. There is thus a moderately robust notion of effective zero-dimensionality, which in one statement is equivalent to computability of the following countable-arity operation $\tilde{S}^X :\subseteq \Sigma^0_1(X)^{\mathsf{N}} \rightrightarrows \Sigma^0_1(X)^{\mathsf{N}}$:

$$\tilde{S}^X((U_i)_i) = \{(W_i)_i \mid (W_i)_i \text{ pairwise disjoint with } W_i \subseteq U_i \text{ and } \bigcup_i W_i = \bigcup_i U_i\}.$$

In this talk, we present a simplified and stronger proof for another result from [1], concerning effectivisation of the following characterisation of zero-dimensionality for nonempty separable metrizable spaces: the space X is zero-dimensional iff every nonempty closed $A \subseteq X$ is a retract of X. Starting from computability of $\tilde{S}^X$ we show

$$E :\subseteq \mathcal{A}(X) \rightrightarrows C(X, X), A \mapsto \{f \mid \operatorname{im} f = A \wedge f|_A = \mathrm{id}_A\}$$

($\operatorname{dom} E = \mathcal{A}(X) \setminus \{\emptyset\}$) is well-defined and computable when the class of closed subsets, $\mathcal{A}(X)$, is represented by $\delta_{\mathrm{range}} \sqcap \delta^{>}_{\mathrm{dist}}$. In our presentation this relates to so-called Dugundji systems for A by pairwise disjoint clopen sets. Moreover, we will elaborate the more general construction of a Dugundji system [5] from a $\delta_{\mathrm{range}} \sqcap \delta^{>}_{\mathrm{dist}}$-name of a closed subset A ($\neq \emptyset, X$) of a computable metric space X, using a computable form of countable paracompactness. Concepts from non-archimedean analysis also lead us to consider the construction of retractions from more general Dugundji systems in certain cases when X is zero-dimensional.

Finally, a converse to the above computability result on E will be discussed (compare [1, Prop 8.5]): if E is computable and X is effectively locally compact in the sense that $X \times (0, \infty) \to \mathcal{K}_{>}(X), (x, r) \mapsto \bar{B}(x; r)$ is well-defined and computable, then X is effectively zero-dimensional. This uses a version of the construction of bilocated sets from the constructive analysis literature, see [4, Ch 7, Prop 4.14].

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DIFFERENTIABILITY AND EFFECTIVE GENERICITY

RUTGER KUYPER

In recent times, several people have studied connections between different notions of randomness and differentiability. These results are all of the form

$x \in [0,1]$ is $\mathcal{A}$-random if and only if every function $f \in \mathcal{C}_{\mathcal{A}}$ is differentiable at x,

where $\mathcal{A}$ is some randomness notion and $\mathcal{C}_{\mathcal{A}}$ is some class of computable functions, which depends on the precise randomness notion being studied. For example, Brattka, Miller and Nies [1], building on work of Demuth, characterised Martin-Löf randomness in this way by taking $\mathcal{C}_{\mathrm{MLR}}$ to be the computable functions of bounded variation, i.e. the functions which are the difference of two non-decreasing functions. Furthermore, they characterised computable randomness by taking $\mathcal{C}_{\mathrm{CR}}$ to be the class of computable non-decreasing functions. Other work in this direction was done by, amongst others, Freer, Kjos-Hanssen, Nies and Stephan [3], Galicki and Turetsky [4], Pathak, Rojas and Simpson [6] and Rute [7].

In this talk we will discuss a similar result for 1-genericity. More precisely, we will show that

$x \in [0,1]$ is 1-generic if and only if every (classically) differentiable, computable function has continuous derivative at x.

This result can be seen as an effectivisation of a result by Bruckner and Leonard [2]. Here, by a differentiable, computable function we mean a function which is computable, and is differentiable, but its derivative need not necessarily be computable.

We will also take a look at ongoing work to characterise n-genericity and polynomial time genericity. For the first, a reasonable guess would be to try and characterise it using n-times differentiable functions. However, we show that

$x \in [0,1]$ is 1-generic if and only if every n-times differentiable, computable function has continuous nth derivative at x.

For the latter, one would guess this could be characterised using differentiable, polynomial time computable functions, but we show that this also corresponds to exactly 1-genericity.

This talk concerns joint work with Sebastiaan Terwijn [5].

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(Rutger Kuyper) Radboud University Nijmegen, Department of Mathematics, P.O. Box 9010, 6500 GL Nijmegen, the Netherlands.
E-mail address: `mail@rutgerkuyper.com`

Game characterizations and Weihrauch degrees

Benedikt Löwe
Institute for Logic, Language, and Computation
University of Amsterdam, Netherlands

Department of Mathematics
University of Hamburg, Germany

Corpus Christi College
University of Cambridge, United Kingdom
b.loewe@uva.nl

Hugo Nobrega
Institute for Logic, Language, and Computation
University of Amsterdam, Netherlands
h.nobrega@uva.nl

Arno Pauly
Clare College
University of Cambridge, United Kingdom
Arno.Pauly@cl.cam.ac.uk

Game characterizations of classes of functions in Baire space $\mathbb{N}^\mathbb{N}$ have a well-established tradition in descriptive set theory, dating back to the work of Wadge and with developments by Duparc, Andretta, Motto Ros, and Semmes, among others (see, e.g., [6] and the references therein). In some cases, these games shed light on otherwise difficult problems and may provide new proofs and indicate generalizations of known theorems (e.g., Semmes's proof and generalizations of the Jayne-Rogers theorem [7]). In particular, this has led to the study of such games in a more abstract setting [5].

In this work we show how such games can be seen in a unified framework with some concepts from computable analysis. We then use this framework to obtain games for the Baire classes of finite rank. Undefined notation and notions are standard, and can be found e.g. in [4] for descriptive set theory and [2] for computable analysis. We denote the computable version of Weihrauch reducibility by $\leq_\mathrm{W}$, and its continuous counterpart by $\leq_\mathrm{W}^\mathrm{c}$.

Definition 1. Let $T :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ and $f :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$. The *$T$-Wadge game for f* is played by two players, **I** and **II**, who take turns in infinitely many rounds. At each round of a run of the game, player **I** first plays a natural number and player **II** then either plays a natural number or passes, as long as she plays natural numbers infinitely often. Therefore, in the long run player **I** builds $x \in \mathbb{N}^\mathbb{N}$ and **II** builds $y \in \mathbb{N}^\mathbb{N}$, and player **II** *wins* the run of the game if $x \notin \mathrm{dom}(f)$, or $y \in \mathrm{dom}(T)$ and $T(y) = f(x)$.

In this abstract setting our main result is the following.

Theorem 2 (Based on results from [3])**.** Every Weihrauch degree has a representative T such that $f \leq_\mathrm{W}^\mathrm{c} T$ ($f \leq_\mathrm{W} T$) iff player **II** has a (computable) winning strategy in the T-Wadge game for f.

The usual game characterizations of function classes in Baire space can be seen as T-Wadge games for appropriate choices of T. For example, the Wadge game is the id-Wadge game, Duparc's eraser game is the lim-Wadge game, and Andretta's backtrack game is the $\lim_\Delta$-Wadge game.

Games for the Baire classes of finite rank

As further concrete examples, we show how one can obtain game characterizations of each Baire class of finite rank using the approach above.

Given a countably-branching tree T, we define its *derivative* T' as the tree obtained by removing from T any node which has a finite bound on the depths of its descendants. Via suitable coding, this can be seen as a map TreeDerivative $:\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$. A *labelled tree* is a tree T equipped with a function $\phi : T \to \mathbb{N}^{<\mathbb{N}}$ which preserves lengths and initial segments.

Proposition 3.

1. The map $\mathrm{label}_{\mathrm{lin}}$ which outputs the labels along the infinite path of a linear labelled tree is computable (where linear means that each node has just one child).
2. The map $\mathrm{label}_{\mathrm{fb}}$ which outputs the labels along the infinite path of a finitely-branching labelled tree with a unique infinite branch is equivalent to lim.
3. TreeDerivative $\equiv_{\mathrm{W}} \lim \star \lim \equiv_{\mathrm{W}} \lim \circ \lim$

We now have the following.

Theorem 4. Let $f :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$. Then

1. Player **II** has a winning strategy in the $\big(\mathrm{label}_{\mathrm{lin}} \circ (\mathrm{TreeDerivative})^n\big)$-Wadge game for f iff f is of Baire class $2n$; and
2. Player **II** has a winning strategy in the $\big(\mathrm{label}_{\mathrm{fb}} \circ (\mathrm{TreeDerivative})^n\big)$-Wadge game for f iff f is of Baire class $2n+1$.

The proof proceeds by showing that $\mathrm{label}_{\mathrm{lin}} \circ (\mathrm{TreeDerivative})^n$ and $\mathrm{label}_{\mathrm{fb}} \circ (\mathrm{TreeDerivative})^n$ satisfy the conditions of Theorem 2 for their Weihrauch degrees, and then by using Proposition 3 and the fact that $\lim^n$ is Weihrauch-complete for Baire class n [1].

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Coherence Spaces for Computable Analysis

Kei Matsumoto and Kazushige Terui
RIMS, Kyoto University
kmtmt@kurims.kyoto-u.ac.jp, terui@kurims.kyoto-u.ac.jp

There have been two mainstream approaches in computable analysis: the type-two theory of effectivity (TTE) and Scott-Ershov domain representations (see [SHT08]). This paper proposes a variant of the second approach based on *coherence spaces*.

Coherence spaces, introduced by Girard [Gi87], are a drastic simplification of Berry's dI-domains. Their distinctive feature is that two kinds of morphism coexist harmoniously: *stable* and *linear* maps. While the former is a domain-theoretic analogue of deterministic, sequential computation (though not fully abstract), the latter is an original discovery that gave birth to linear logic.

We import the concept of *admissible representation* from TTE, and provide admissible representations for the real line $\mathbb{R}$, Euclidean spaces $\mathbb{R}^n$ and function spaces $\mathcal{C}(\mathbb{R}^n, \mathbb{R})$ among others. This allows us to represent, for instance, a real continuous operator $\Phi : \mathcal{C}(\mathbb{R}^n, \mathbb{R}) \longrightarrow \mathcal{C}(\mathbb{R}^n, \mathbb{R})$ by a *stable* map.

A natural question is then what *linear* maps correspond to. Our partial answer is that they correspond to *uniformly* continuous functions, when we focus on the functions from $\mathbb{R}$ to $\mathbb{R}$ and choose a specific representation of $\mathbb{R}$. This leads to an internal expression of *Heine's theorem* (every continuous function on a compact interval $\mathbb{I} \subseteq \mathbb{R}$ is uniformly continuous) as the existence of a certain stable (and computable) map from a stable function space to a linear function space.

A *coherence space* $X = (|X|, \frown\!\!\!\!\smile)$ consists of a set $|X|$ of *tokens* and a reflexive symmetric relation $\frown\!\!\!\!\smile$ on $|X|$, called *coherence*. The relation $\frown\!\!\!\!\smile$ is often obtained by taking the reflexive closure of an irreflexive symmetric relation $\frown$. A *clique* of X is a set of pairwise coherent tokens in $|X|$. The set of cliques (resp. finite cliques, maximal cliques) is denoted by $\mathsf{Clq}(X)$ (resp. $\mathsf{FClq}(X)$, $\mathsf{MClq}(X)$). The ordered set $(\mathsf{Clq}(X), \subseteq)$ forms a Scott domain, so is equipped with the *Scott topology* generated by $\langle a \rangle := \{b \in \mathsf{Clq}(X) \mid a \subseteq b\}$ for all $a \in \mathsf{FClq}(X)$ as a base.

Let X and Y be coherence spaces. A *stable map* $F : X \longrightarrow_{st} Y$ is a continuous function from $\mathsf{Clq}(X)$ to $\mathsf{Clq}(Y)$ such that for any cliques $a, b \in \mathsf{Clq}(X)$, $a \cup b \in \mathsf{Clq}(X)$ implies $F(a \cap b) = F(a) \cap F(b)$. On the other hand, a *linear map* $F : X \longrightarrow_{lin} Y$ is a function such that $F(\sum_i a_i) = \sum_i F(a_i)$, where $\sum$ means disjoint union of cliques. Linear maps are always stable.

Given coherence spaces X_1, X_2, the product $X_1 \times X_2$, the stable function space $X_1 \Rightarrow X_2$ and the linear function space $X_1 \multimap X_2$ are naturally defined. It is well known that the category **Coh** of coherence spaces and stable maps is cartesian closed, while the category **Lin** of coherence spaces and linear maps is monoidal closed, giving rise to a *Seely category*, a categorical model of linear logic (see [Me09]).

Let $\mathbb{X}$ be an arbitrary set. A *representation* is a triple $X \xrightarrow{\rho_X} \mathbb{X}$, where X is a coherence space and $\rho_X :\subseteq \mathsf{Clq}(X) \longrightarrow \mathbb{X}$ is a partial surjective function. A *relatively stable map* from $X \xrightarrow{\rho_X} \mathbb{X}$ to $Y \xrightarrow{\rho_Y} \mathbb{Y}$ is a total function $f : \mathbb{X} \longrightarrow \mathbb{Y}$ which is *realized* by a stable map $F : X \longrightarrow_{st} Y$, i.e., $f \circ \rho_X = \rho_Y \circ F$ holds on $\mathsf{dom}(\rho_X)$:

$$
\begin{array}{ccc}
\mathsf{Clq}(X) & \xrightarrow{\quad F \quad} & \mathsf{Clq}(Y) \\
\Big\downarrow \rho_X & & \Big\downarrow \rho_Y \\
\mathbb{X} & \xrightarrow{\quad f \quad} & \mathbb{Y}
\end{array} \qquad . \tag{1}
$$

The notion of *relatively linear map* is similarly defined. The cartesian closed structure of **Coh** naturally carries over to the category **CohRep** of representations and relatively stable maps. We denote an exponential as $X \Rightarrow Y \overset{[\rho_X \Rightarrow \rho_Y]}{\longrightarrow} \mathbb{Y}^{\mathbb{X}}$ and the n-th power as $X^n \overset{\rho_X^n}{\longrightarrow} \mathbb{X}^n$.

Let us introduce a canonical representation of the real line $\mathbb{R}$. Let $\mathbb{D} := \mathbb{Z} \times \mathbb{N}$, where each $(m, n) \in \mathbb{D}$ is identified with a dyadic rational number $m/2^n$. The coherence space $\mathbf{R}$ is defined to be $(\mathbb{D}, \frown\!\!\!\!\smile)$, where for every $x = (i, j)$ and $y = (k, l)$ in $\mathbb{D}$,

$$x \frown y \iff j \neq l \text{ and } |x - y| \leqslant 2^{-j} + 2^{-l}.$$

It is then easy to see that a maximal clique $a \in \mathsf{MClq}(\mathbf{R})$ expresses a Cauchy sequence $\{x_n\}_{n \in \mathbb{N}}$. Thus we may define a partial surjective function $\rho_{\mathbf{R}}(a) := \lim x_n$ with $\mathsf{dom}(\rho_{\mathbf{R}}) = \mathsf{MClq}(\mathbf{R})$, that constitutes a representation $\mathbf{R} \overset{\rho_{\mathbf{R}}}{\longrightarrow} \mathbb{R}$.

We now address the issue of admissibility. Let $T \subseteq \mathsf{FClq}(\mathbf{R})$ be the set of finite cliques that correspond to *initial segments* of Cauchy sequences. Then we have either $a \subseteq b$ or $b \subseteq a$ for any $a, b \in T$ whenever $a \cup b \in \mathsf{Clq}(\mathbf{R})$. Moreover, every maximal clique (Cauchy sequence) is a limit of cliques in T. In other words, the set $\mathsf{MClq}(\mathbf{R})$ is approximated by the tree T of finite cliques, just as the Cantor space Σ^ω is approximated by Σ^*. This motivates the following definition.

Let $\mathbb{X}$ be a topological space. A partial function $\gamma :\subseteq \mathsf{Clq}(X) \longrightarrow \mathbb{X}$ is called *pre-admissible* if it is continuous and there exists a set of finite cliques $T \subseteq \mathsf{FClq}(X)$ such that

- for any $a, b \in T$ with $a \cup b \in \mathsf{Clq}(X)$, either $a \subseteq b$ or $b \subseteq a$ holds;
- $a \in \mathsf{dom}(\gamma)$ iff there is a maximal chain $\{a_i\}_{i \in I}$ in T such that $a = \bigcup a_i$.

A representation $Y \overset{\rho_Y}{\longrightarrow} \mathbb{Y}$ is *admissible* if ρ_Y is pre-admissible, and moreover for any pre-admissible function $\gamma :\subseteq \mathsf{Clq}(X) \longrightarrow \mathbb{Y}$ (that need not be surjective), there exists a stable map $F : X \longrightarrow_{st} Y$ such that $\gamma = \rho_Y \circ F$ holds on $\mathsf{dom}(\gamma)$. The crux of admissibility is the following fact (cf. [Sc02]).

Theorem 1 *Let $X \overset{\rho_X}{\longrightarrow} \mathbb{X}$ and $Y \overset{\rho_Y}{\longrightarrow} \mathbb{Y}$ be admissible representations. A function $f : \mathbb{X} \longrightarrow \mathbb{Y}$ is sequentially continuous if and only if it is relatively stable (i.e., there is a stable map $F : X \longrightarrow_{st} Y$ that makes Diagram (1) commute).*

As expected, the representation $\mathbf{R} \overset{\rho_{\mathbf{R}}}{\longrightarrow} \mathbb{R}$ and its functional extensions are all admissible.

Theorem 2 *$\mathbf{R} \overset{\rho_{\mathbf{R}}}{\longrightarrow} \mathbb{R}$ is admissible. Moreover, $\mathbf{R}^n \Rightarrow \mathbf{R} \overset{[\rho_{\mathbf{R}}^n \Rightarrow \rho_{\mathbf{R}}]}{\longrightarrow} \mathcal{C}(\mathbb{R}^n, \mathbb{R})$ is admissible for every $n \geqslant 1$, where the function space $\mathcal{C}(\mathbb{R}^n, \mathbb{R})$ is equipped with the compact-open topology.*

As a consequence, a real operator $\Phi : \mathcal{C}(\mathbb{R}^n, \mathbb{R}) \longrightarrow \mathcal{C}(\mathbb{R}^n, \mathbb{R})$ is sequentially continuous if and only if it is realized by a stable map.

The above theorem is not as obvious as it may look, since stability is a stronger constraint than mere Scott continuity. It should be recalled that stability is proposed as a better approximation of deterministic, sequential computation. Hence refining a Scott continuous map to a stable one is somewhat similar to *determinizing* a nondeterministic Turing machine, that turns out tricky in our denotational setting. It is our peculiar definition of pre-admissibility that makes everything work. We also remark that a topological space is admissibly representable in the sense of TTE if and only if it is so in our sense.

So far we have just traced the development that occurred in TTE and domain representations. We now proceed to an original aspect of coherence spaces: the notion of linearity. It turns out that our specific representation $\mathbf{R} \xrightarrow{\rho_{\mathbf{R}}} \mathbb{R}$ implies a curious fact.

Theorem 3 *A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is relatively linear if and only if it is uniformly continuous.*

This allows us to internally express *Heine's theorem* stating that every continuous function $f : \mathbb{I} \longrightarrow \mathbb{R}$, where $\mathbb{I} \subseteq \mathbb{R}$ is a compact interval, is uniformly continuous. Let us write $\mathcal{C}_u(\mathbb{I}, \mathbb{R})$ for the space of uniformly continuous real functions. We obviously have $\mathcal{C}_u(\mathbb{I}, \mathbb{R}) \subseteq \mathcal{C}(\mathbb{I}, \mathbb{R})$, while Heine's theorem states the converse.

Let $\mathbf{I}$ be the restriction of the coherence space $\mathbf{R}$ that corresponds to the interval $\mathbb{I}$. By the previous theorem, there exists a representation $\mathbf{I} \multimap \mathbf{R} \xrightarrow{[\rho_{\mathbf{I}} \multimap \rho_{\mathbf{R}}]} \mathcal{C}_u(\mathbb{I}, \mathbb{R})$, where $[\rho_{\mathbf{I}} \multimap \rho_{\mathbf{R}}]$ is a linear analogue of $[\rho_{\mathbf{I}} \Rightarrow \rho_{\mathbf{R}}]$. We then have:

Theorem 4 *There exists a stable map $\mathcal{H} : (\mathbf{I} \Rightarrow \mathbf{R}) \longrightarrow_{st} (\mathbf{I} \multimap \mathbf{R})$ that satisfies:*

$$
\begin{array}{ccc}
\mathbf{I} \Rightarrow \mathbf{R} & \xrightarrow{\quad \mathcal{H} \quad} & \mathbf{I} \multimap \mathbf{R} \\
{\scriptstyle [\rho_{\mathbf{I}} \Rightarrow \rho_{\mathbf{R}}]} \downarrow & & \downarrow {\scriptstyle [\rho_{\mathbf{I}} \multimap \rho_{\mathbf{R}}]} \\
\mathcal{C}(\mathbb{I}, \mathbb{R}) & \xrightarrow{\quad id \quad} & \mathcal{C}_u(\mathbb{I}, \mathbb{R})
\end{array}
\quad .
$$

The map $\mathcal{H}$ is in fact computable. The idea is first to compute the "modulus of continuity" of a given stable map $F : \mathbf{I} \longrightarrow_{st} \mathbf{R}$, and then to transform F into a linear map $\mathcal{H}(F) : \mathbf{I} \longrightarrow_{lin} \mathbf{R}$ by employing the "modulus" information.

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Continuous enclosures and best approximations of discontinuous operations

Eike Neumann
Aston University, Birmingham, United Kingdom
neumaef1@aston.ac.uk

Many practically relevant problems encountered in continuous mathematics, such as global optimisation, finding zeroes of continuous functions on the unit interval, finding fixed points of continuous mappings, solving general ordinary differential equations, or finding an eigenbasis of a potentially singular symmetric matrix are unsolvable in general because their respective solution operations are discontinuous. To allow for a computable solution one either has to place further constraints on the admissible problem instances or assume further information on the input to be available.

Here, we would like to attempt a different approach: find a generic algorithm that works on all inputs without any extra information and outputs as much information as possible on the solution. Our main question is thus:

Given an operation f, what is the largest amount of information on f that can be continuously obtained?

To formalise this question, let us first specify what we mean by "partial information" on a mapping. If X is a represented space, we denote by $\mathscr{K}(X)$ and $\mathscr{V}(X)$ the spaces of saturated compact subsets and overt subsets of X, as defined in [6], respectively.

Definition 1. Let X and Y be represented spaces and let $f\colon X \rightrightarrows Y$ be an operation. A set-valued mapping $F\colon X \to \mathscr{V}(\mathscr{K}(Y))$ is called an *enclosure* of f if for every $x \in X$ and every $K \in F(x)$ there exists a $y \in f(x)$ such that $y \in K$.

A saturated compact set $K \in \mathscr{K}(Y)$ can be identified with the open set of all open sets containing K. Thus, K can be viewed as an effective encoding of the observable properties which are shared by all its members. Recall that if Y is admissible, then a point can be effectively recovered from the collection of all its observable properties. Consequently, a compact set effectively encodes partial information on every point it contains.

If $\mathscr{K}(Y)$ is *selectable* (see [1, Definition 5.2 and Theorem 5.4]), e.g. if Y is a complete separable metric space (see [1, Corollary 6.4] for a similar result), then a continuous set-valued mapping $F\colon X \to \mathscr{V}(\mathscr{K}(Y))$ is continuous as an operation $F\colon X \rightrightarrows Y$, i.e. it has a continuous realiser. Furthermore, it satisfies the following realisability property: Let $\delta_X\colon \subseteq \mathbb{N}^{\mathbb{N}} \to X$ and $\kappa_Y\colon \subseteq \mathbb{N}^{\mathbb{N}} \to \mathscr{K}(Y)$ denote the canonical representations of X and $\mathscr{K}(Y)$ respectively. For all $x \in X$ and all sets $K \in F(x)$ there exist a δ_X-name $p_0 \in \mathbb{N}^{\mathbb{N}}$ of x, and a continuous partial mapping on Baire space $R\colon \subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ with $\kappa_Y(R(p)) \in F(\delta_X(p))$ for all $p \in \mathbb{N}^{\mathbb{N}}$ and $\kappa_Y(R(p_0)) = K$.

Single-valued enclosures, i.e. enclosures mapping every point to the closure of a singleton $\overline{\{K\}} \in \mathscr{V}(\mathscr{K}(Y))$ can be effectively identified with single-valued mappings sending X to $\mathscr{K}(Y)$ and correspond topologically to upper hemi-continuous functions.

The next task is to provide a means for comparing enclosures with respect to the amount of information they contain. There are two natural orders we can consider.

Definition 2. Let $F\colon X \to \mathscr{V}(\mathscr{K}(Y))$ and $G\colon X \to \mathscr{V}(\mathscr{K}(Y))$ be set-valued mappings. We say that F *tightens* G (G *loosens* F) if for every $x \in X$ and every $K \in G(x)$, there is an $L \in F(x)$ such that $K \supseteq L$. We say that F *narrows* G (G *widens* F) if for every $x \in X$ and every inclusion-minimal $L \in F(x)$, there is an inclusion-minimal $K \in G(x)$ with $K \supseteq L$.

The tightening- and narrowing-relations are partial orders on the space of continuous mappings sending X to $\mathscr{V}(\mathscr{K}(Y))$.

Note that if we identify the operation $f\colon X \rightrightarrows Y$ with a single-valued mapping sending X to $\mathscr{V}(\mathscr{K}(Y))$, then F is an enclosure of f if and only if it is tightened by f.

A major motivating example is the problem of solving nonlinear equations on the unit interval. Consider the solution operation:

$$\begin{cases} \textsc{zero}\colon \subseteq \mathrm{C}([0,1]) \rightrightarrows [0,1], \\ \mathrm{dom}(\textsc{zero}) = \{f\colon [0,1] \to \mathbb{R} \mid f(0)\cdot f(1) < 0\}, \\ \textsc{zero}(f) = \{x \in [0,1] \mid f(x) = 0\}. \end{cases}$$

It is well known (and easy to see) that this operation is discontinuous. We can however obtain a rather nice continuous enclosure. Following [3], we call a component C of the zero set $\mathscr{Z}(f)$ of f *robust* if on every neighbourhood of C every sufficiently small perturbation of f still has a zero. Note that the robust components of f are the closure (in the Hausdorff metric) of those components of $\mathscr{Z}(f)$ where f "changes sign". The tightest continuous enclosure of ZERO is given by:

$$\begin{aligned} &E\colon \mathrm{dom}(\textsc{zero}) \to \mathscr{V}(\mathscr{K}([0,1])) \\ &E(f) = \mathrm{closure}\left(\{K \subseteq [0,1] \mid K \text{ is a robust connected component of } \mathscr{Z}(f)\}\right). \end{aligned}$$

This enclosure is equivalent to the mapping which sends a function to the *component cover representation* of its robust zero set, introduced by Collins [3]. The fact that it is the tightest enclosure shows that this representation - in some sense - encodes the largest amount of "positive information" that can be continuously obtained on the zero set. One can show that there is no narrowest continuous enclosure, and in fact, no maximally narrow continuous enclosure of ZERO.

Using index theory, as laid out in [3], this example generalises to higher dimensions and yields a tightest continuous enclosure of the solution operator for nonlinear fixed point equations. Similar techniques which implicitly yield this result are used in [5] and [2].

Generally, it is quite easy to find the tightest enclosure for sufficiently well-understood continuous problems with discontinuous solution operation, such as global optimisation, solving linear equations, finding fixed points of nonexpansive mappings, or finding eigenbases of potentially singular symmetric matrices.

Under reasonable further assumptions on Y and f, for instance if Y is a compact metric space and $f(x)$ is a closed subset of Y for every $x \in X$, a tightest enclosure $F\colon X \to \mathscr{V}(\mathscr{K}(Y))$ of f can be shown to always exist. However, it is a subtle question how well this enclosure can be realised as an operation $F\colon X \rightrightarrows Y$. This is a subject of ongoing research.

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Computability on the countable ordinals and the Hausdorff-Kuratowski theorem

Arno Pauly
Clare College
University of Cambridge, United Kingdom
Arno.Pauly@cl.cam.ac.uk

This note continues a research programme to investigate concepts from descriptive set theory in the very general setting of represented spaces, and in a fashion that produces both classical and effective results simultaneously. A survey of this approach is given in [8]. One of the first theorems studied in this way is the Jayne-Rogers theorem ([4], simplified proof in [7]); a computable version holding also in some non-Hausdorff spaces was proven by the author and de Brecht in [10] using results about Weihrauch reducibility in [1].

Our goal for this note is to state and prove a corresponding version of the Hausdorff-Kuratowski theorem. For this, we require a notion of computability on the space of countable ordinals – and such a theory would be foundational for several further results in the research programme. Apart from some initial investigations in [6], there is no established definition of a computability structure on the countable ordinals. There is, of course, a well-established notion of what a computable ordinal is, however, this does not suffice for our purposes. We will investigate some promising candidates, and suggest one equivalence class as the standard to be adopted:

Definition 1. Let COrd be the set of countable ordinals. We define $\delta_{\mathrm{nK}} :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathrm{COrd}$ inductively via:

1. $\delta_{\mathrm{nK}}(0p) = 0$
2. $\delta_{\mathrm{nK}}(1p) = \delta_{\mathrm{nK}}(p) + 1$
3. $\delta_{\mathrm{nK}}(2\langle p_0, p_1, p_2, \ldots\rangle) = \sup_{i\in\mathbb{N}} \delta_{\mathrm{nK}}(p_i)$.

Definition 2. We will consider the equivalence class of δ_{nK} as the standard representation of COrd, and thus abbreviate $\mathbf{COrd} := (\mathrm{COrd}, \delta_{\mathrm{nK}})$.

Roughly following [5, Section 22.E], we shall recall the definition of the difference hierarchy. We define a function par from the countable ordinals to $\{0, 1\}$ by $\mathrm{par}(\alpha) = 0$, if there is a limit ordinal β and a number $n \in \mathbb{N}$ such that $\alpha = \beta + 2n$; and $\mathrm{par}(\alpha) = 1$ otherwise. For a fixed ordinal α, we let $\mathfrak{D}_\alpha(\mathbb{N}^{\mathbb{N}})$ be the collection of sets $D \subseteq \mathbb{N}^{\mathbb{N}}$ definable in terms of a family $(U_\lambda)_{\lambda<\alpha}$ of open sets via:

$$x \in D \Leftrightarrow \mathrm{par}\left(\inf\{\beta \mid x \in U_\beta\}\right) \neq \mathrm{par}(\alpha)$$

Theorem 3 ((Classical) Hausdorff-Kuratowski theorem)**.** $\boldsymbol{\Delta}^0_2(\mathbb{N}^{\mathbb{N}}) = \bigcup_{\alpha\in\mathrm{COrd}} \mathfrak{D}_\alpha(\mathbb{N}^{\mathbb{N}})$

The theorem extends to functions in a rather straight-forward way, based on the *level* introduced by Hertling:

Definition 4 (Hertling [3])**.** Given a function $f :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$, we define the sets $\mathcal{L}_\alpha(f) \subseteq \mathbb{N}^{\mathbb{N}}$ inductively via:

1. $\mathcal{L}_0(f) = \mathrm{dom}(f)$
2. $\mathcal{L}_{\alpha+1}(f) = \overline{\{x \in \mathcal{L}_\alpha(f) \mid f|_{\mathcal{L}_\alpha} \text{ is discontinuous at } x\}}$
3. $\mathcal{L}_\gamma(f) = \bigcap_{\beta<\gamma} \mathcal{L}_\beta(f)$ for limit ordinals γ.

Then we say $\mathrm{Lev}(f) := \min\{\alpha \mid \mathcal{L}_\alpha(f) = \emptyset\}$.

Theorem 5 (Hausdorff-Kuratowski theorem for functions, DE BRECHT [2])**.** A function $f : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ is $\utilde{\mathbf{\Delta}}^0_2$-measurable iff $\mathrm{Lev}(f)$ exists.

In order to formulate a uniform version, we introduce a family of endofunctors $\mathfrak{L}_{(\alpha_i)_{i\in\mathbb{N}}}$ capturing computation with finitely many mindchanges indexed by the family of ordinals $(\alpha_i)_{i\in\mathbb{N}}$. This makes a function f realizable w.r.t. $\mathfrak{L}_{(\alpha_i)_{i\in\mathbb{N}}}$ iff $\mathrm{Lev}(f) \leq \sup_{i\in\mathbb{N}} \alpha_i$. We will also need dependent sum types for represented spaces. With this in place, we can then fulfill our original goal:

Theorem 6 (Computable Hausdorff-Kuratowski theorem)**.** Let $\mathbf{X}$, $\mathbf{Y}$ be represented spaces, and $\mathbf{X}$ be complete. Then the map $\mathrm{HK} : \mathcal{C}(\mathbf{X}, \mathbf{Y}^\nabla) \rightrightarrows \sum_{(\alpha_i)_{i\in\mathbb{N}} \in \mathbf{COrd}^\mathbb{N}} \left(\mathcal{C}(\mathbf{X}, \mathfrak{L}_{(\alpha_i)_{i\in\mathbb{N}}}\mathbf{Y})\right)$ where $((\alpha_i)_{i\in\mathbb{N}}, g) \in \mathrm{HK}(f)$ iff $f = g$, is computable and has a computable retract.

An important corollary of this was already established by SELIVANOV:

Corollary 7. Let $f : \mathbf{X} \to \mathbf{Y}$ be computable with finitely many mindchanges, and $\mathbf{X}$ be complete. Then $\mathrm{Lev}(f)$ exists and is a computable ordinal.

A full draft is available on the arXiv as [9].

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CafeOBJ for real – using an algebraic specification language as theorem prover for computational reals

Norbert Preining

Japan Advanced Institute of Science and Technology
Research Center for Software Verification
Nomi, Ishikawa, Japan
preining@jaist.ac.jp

CafeOBJ is a many- and order-sorted algebraic specification language from the OBJ family, related to languages like CASL and MAUDE. CafeOBJ allows us to have both the specification and the verification in the same language. It is based on powerful logical foundations (order-sorted algebra, hidden algebra, and rewriting logic) with an executable semantics [4, 6, 7]. It has been used to specify and verify extensive systems (e.g., railway signal systems, e-commerce protocols, authentication protocols), but at the same time it functions equally well as theorem prover using the built-in rewrite engine.

On the other hand, provable properties of computations on reals are highly desirable, and computational models of the reals based on sound logical foundations with effective algorithm form a favorable approach. Computational models of the reals are wide, ranging from interval arithmetic to all kind of sequence approximations, in particular streams (signed digit streams etc). Representing streams in a formal framework requires extra caution, as streams are infinite objects. Typical approaches use coinductive definitions (e.g., [1, 5]).

Stream processing [2, 3] is a specific case of behavioral rewrite systems [11], and thus can be implemented in CafeOBJ, which provides *hidden sorts* allowing us to specify infinite data objects via a behavioral, i.e. coinductive, approach.

Aim of this work is a re-interpretation of CafeOBJ's 'specification' part to provide an 'implementation' of exact reals, and its 'verification' part to prove properties of the exact reals. Verification is realized by an executable semantics based on order-sorted rewriting. Using this approach, we provide the implementation, as well as its the verification of adequateness, as well as proofs of properties, all in the same framework of CafeOBJ.

After giving a short introduction to CafeOBJ and its syntax and semantical background, we will touch upon three areas related to exact real representation: (1) Extension of the number tower of CafeOBJ; (2) implement modules for the exact representation of reals; (3) use this representation to prove some simple lemmas.

ad 1. CafeOBJ ships modules for nearly the whole number stack, based on the underlying Lisp implementation. In some cases, these modules don't use inheritance (algebraic extension) with the effect that number overloading and combining of numbers of different types is not possible. We fixed these insufficiencies and expect the changes to become part of the main CafeOBJ distribution.

ad 2. By representing signed digit streams as hidden sorts, i.e., co-algebraically, we obtained a representation module of exact reals. We also developed conversion functions to translate between signed digit streams and floats. Furthermore, additional operations to deal with signed digit streams are provided.

ad 3. Using this representation obtained, we carry out some simple proofs, similar to [1, 5], concerning arithmetical operations and average functions.

Relation to other implementations There are for sure more powerful and complete implementations of exact reals in other languages, like Coq [9], Haskell [10], or C++ [8]. We see the advantage of using CafeOBJ in its equational reasoning. First-order expressions have to be explicitly skolemized, and the necessary properties for Skolem functions have to be added in the course of proving, which requires explicit constructions.

This means that developing a proof in CafeOBJ, an inherently interactive process, incorporates the extraction of computational content from the proof. Berger [1] states that "[...] *program extraction from proofs has turned out to be very reliable and useful methodology for obtaining certified proofs, even if the extraction is done with pen and paper* [...]." By executing intermediate stages of the proof, CafeOBJ guides the user to insufficient information and missing lemmata, and thus can play a supporting role in the extraction of computational content.

We plan to extend the current code-base to cover at least all standard arithmetic functions including trigonometric ones, and cover similar topics as presented in the references.

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A Hofmann-Mislove Theorem for Scott open sets

Matthias Schröder, TU Darmstadt

The Hofmann-Mislove Theorem states that in a sober topological space X the intersection map $\mathcal{F} \mapsto \bigcap(\mathcal{F})$ is a bijection between the non-empty Scott-open filters on the lattice of open subsets of X and the compact saturated subsets of X [3].

The natural question arises how to generalise this result to all non-empty Scott-open families of opens. We give a positive answer in the case of sequentially Hausdorff sequential spaces. Remember that a space is called *sequentially Hausdorff*, if every convergent sequence has a unique limit. Our version of the Hofmann-Mislove Theorem states that the intersection map defined on the family $\mathcal{OO}_+(X)$ of all non-empty ω-Scott-open collections of opens is a continuous retraction onto the space $\mathcal{K}(X)$ of all countably-compact subsets in such spaces X. The corresponding topologies are the ω-Scott topology on $\mathcal{OO}_+(X)$ and the sequentialisation of the upper Vietoris topology on $\mathcal{K}(X)$. Both are natural topologies in the category of sequential spaces. For the proof we only use Dependent Choice (DC), whereas the classical Hofmann-Mislove Theorem needs the Axiom of Choice (AC).

Sequential spaces and qcb-spaces

A *sequential space* is a topological space in which every sequentially open subset is open [2]. The category Seq of sequential spaces and continuous functions is cartesian closed [1]. A *qcb-space* is a quotient of a countably based space. Qcb-spaces play an important role in Computable Analysis: they form the class of sequential spaces which can be appropriately handled by Weihrauch's Type Two Model of Effectivity (TTE) [5]. The category QCB of qcb-spaces forms a cartesian closed subcategory of Seq.

The Scott topology on opens and the upper Vietoris topology on compacts

Let X be a sequential space. We equip the family $O(X)$ of open subsets of X with the ω-Scott topology on the complete lattice $(O(X); \subseteq)$ and denote this space by $\mathcal{O}(X)$. Remember that a subset $H \subseteq \mathcal{O}(X)$ is *ω-Scott open*, if H is upwards closed and $\mathcal{D} \cap H \neq \emptyset$ holds for each countable directed subset $\mathcal{D}$ with $\bigcup \mathcal{D} \in H$. The reason for which we prefer the ω-Scott-topology over the more familiar Scott-topology on $(O(X); \subseteq)$ is that the former is always sequential. Hence $\mathcal{O}(X)$ is an element of Seq. The space $\mathcal{O}(\mathcal{O}(X))$ has the ω-Scott-open families of opens of X as underlying set; its topology is ω-Scott-topology on complete lattice $(O(\mathcal{O}(X)); \subseteq)$. If X is a qcb$_0$-space then the Scott topology coincides with the ω-Scott-topology and both $\mathcal{O}(X)$ and $\mathcal{O}(\mathcal{O}(X))$ are a qcb$_0$-spaces.

A subset K of X is called *countably compact*, if every countable open cover of K has a finite subcover [2]. We equip the set of countably compact subsets of X with the sequentialisation of the upper Vietoris topology. The *upper Vietoris topology* is generated by the subbasic open sets $\Box U := \{K \text{ countably compact} \mid K \subseteq U\}$, where U varies over the open subsets of X. By $\mathcal{K}(X)$ we denote the space of countably compact subsets of X equipped with the sequentialisation of the upper Vietoris topology. If X is T_1, then $\mathcal{K}(X)$ is a T_0-space; if X is a qcb-space, then $\mathcal{K}(X)$ is a qcb-space as well.

Sequentially Hausdorff sequential spaces

A topological space is called *sequentially Hausdorff*, if any convergent sequence has a unique limit. Hausdorff spaces are sequentially Hausdorff and sequentially Hausdorff spaces are T_1. Sequentially Hausdorff spaces are characterised by the property that the inequality test as a map into the Sierpiński space $\mathbb{S}$ is continuous, and also by the property that the map $x \mapsto X \setminus \{x\} \in \mathcal{O}(X)$ is continuous.

The main result

By $\mathcal{OO}_+(X)$ we denote the subspace of $\mathcal{O}(\mathcal{O}(X))$ consisting of all non-empty ω-Scott open families of open sets of X. This space is sequential by being open in $\mathcal{O}(\mathcal{O}(X))$.

Theorem 1 *Let X be a sequentially Hausdorff sequential space. Then the space $\mathcal{K}(X)$ is a continuous retract of $\mathcal{OO}_+(X)$. The map $H \mapsto \bigcap(H)$ ($H \in \mathcal{OO}_+(X)$) is a continuous retraction to the continuous section $K \mapsto \{U \in \mathcal{O}(X) \mid K \subseteq U\}$.*

The proof is based on properties of convergent sequences in the spaces $\mathcal{O}(X)$, $\mathcal{OO}_+(X)$ and $\mathcal{K}(X)$, see [4] for details. The condition of X being sequentially Hausdorff is essential, otherwise $\bigcap(H)$ might not even be countably compact. Theorem 1 implies:

Corollary 2 *Let X be a sober sequentially Hausdorff sequential space. Then the map*

$$H \mapsto \big\{U \in \mathcal{O}(X) \;\big|\; \bigcap(H) \subseteq U\big\}$$

is a continuous function from the family of all non-empty ω-Scott-open subsets of $\mathcal{O}(X)$ to the family of all non-empty ω-Scott-open filters of $\mathcal{O}(X)$.

Corollary 3 *Let X be a sequentially Hausdorff qcb-space. Then $\bigcap(H)$ is compact for every non-empty Scott-open subset $H \subseteq \mathcal{O}(X)$.*

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Average-Case Complexity of Real Functions*

Matthias Schröder, Florian Steinberg , and Martin Ziegler, TU Darmstadt

[1] **Abstract.** We introduce, and initiate the study of, average-case complexity theory over the reals.

In worst-case analyses of algorithms, rare instances may dominate the complexity of an otherwise easy problem. For example Hoare's `QuickSort` takes time quadratic in N in the worst-case but, with respect to the uniform distribution on $\{1,\ldots,N\}$, only $\mathcal{O}(N\log N)$ steps — in agreement with practical experience of usually being a highly efficient method. In general the 'appropriate' distribution for an average-case analysis might be subject to debates, though. For instance regarding floating-point calculations, which one should be considered on the rational unit interval $[0;1]\cap\mathbb{Q}$? The real unit interval on the other hand does come with a canonical probability measure, thus conceptionally simplifying matters compared to the discrete case! We introduce the corresponding formal notions and demonstrate the gap between worst-case and average-case analyses. Note that the latter amounts to the expected deterministic running time on random inputs, not to probabilistic computations [Boss08, BHG13].

Definition 1. *a) Let $\mathcal{M}$ denote a Type-2 Machine computing a total real function $f:[0;1]\to\mathbb{R}$ in the sense of producing, given an integer sequence (a_m) encoded in binary*[†] *with $|x-a_m/2^m|\leq 2^{-m}$ for some $x\in[0;1]$, an integer sequence (b_n) encoded in binary with $|f(x)-b_n/2^n|\leq 2^{-n}$. Let $T_{\mathcal{M}}\big((a_m),n\big)$ denote the number of steps $\mathcal{M}$ makes on input (a_m) before producing b_n; and, for fixed $x\in\mathrm{dom}(f)$,*

$$T_{\mathcal{M}}(x,n) \quad:=\quad \max\left\{T_{\mathcal{M}}\big((a_m),n\big)\;\middle|\;\forall m: |x-a_m/2^m|\leq 2^{-m}\right\}$$

the local runtime in the worst-case over all sequences of input approximations to this x.

b) The (worst-case) time complexity *of f is* polynomial *if there exists such a machine $\mathcal{M}$ with $\sup_{x\in\mathrm{dom}(f)}T_{\mathcal{M}}(x,n)\leq p(n)$ for some polynomial $p\in\mathbb{N}[N]$; equivalently: if there exists some $\varepsilon>0$ such that $\frac{1}{n}\sup_{x\in\mathrm{dom}(f)}\big(T_{\mathcal{M}}(x,n)\big)^{\varepsilon}$ is bounded indepently of n.*

c) Fix a (not necessarily computable) probability measure space (X,Σ,P). A measurable function $F:X\times\mathbb{N}\to[0;\infty]$ is said to be naïvely polynomial on average *if $\bar{F}(n):=\int_X F(x,n)\,dP(x)\leq p(n)$ for some polynomial $p\in\mathbb{N}[N]$.*
F is (non-naïvely) polynomial on average *if if there exists some $\varepsilon>0$ such that $\frac{1}{n}\int_X F(x,n)^{\varepsilon}\,dP(x)$ is bounded independently of n.*

Combining Items a) and c), observe how both naïvely and non-naïvely polynomial running time on average arise from equivalent characterizations of polynomial worst-case bounds (b). However, naïve polynomial average time complexity is not robust under quadratic slow-down [Gold97, §A]; hence we focus in the sequel on the non-naïve notion, underlying also Leonid Levin's structural average-case complexity theory [BoTr06, §2.2.1].

Proposition 2. *a) For $\mathcal{M}$ computing $f:[0;1]\to\mathbb{R}$ according to Definition 1a), $T_{\mathcal{M}}(x,n)$ is well-defined and measurable.*

b) Naïvely polynomial on average implies polynomial on average, but not vice versa.

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†each element without leading zeros, separated by some delimiter

c) *Measurable* $F : X \times \mathbb{N} \to [0;\infty]$ *is polynomial on average iff there exists some* $\delta > 0$ *such that* $P(\{x : F(x,n) \geq t\}) \leq p(n)/t^\delta$ *holds for some polynomial* $p \in \mathbb{N}[N]$*, all* $n \in \mathbb{N}$*, and all* $t > 0$.

d) *Suppose* $f : [0;1] \to \mathbb{R}$ *is computed by* $\mathcal{M}$*. Then* $\mu(x,n) := T_{\mathcal{M}}(x,n+1)+1$ *is a local modulus of continuity in the sense of satisfying* $\forall x' \in [0;1] : |x-x'| \leq 2^{-\mu(x,n)} \Rightarrow |f(x)-f(x')| \leq 2^{-n}$*; compare* [Ko91, Theorem 2.19].

We first refine [KMRZ12, Example 1.12]:

Theorem 3. a) *The function*

$$f : [0;1] \to [0;1] \quad \textit{with} \quad f(0)=0 \quad \textit{and} \quad f(x) = 1/\ln(e/x) = \tfrac{1}{1-\ln x} \textit{ otherwise}$$

is computable in exponential time but, admitting no subexponential global modulus of continuity, not faster, even relative to any oracle — in the worst case: Its (naïve) average-time complexity is polynomial; in fact it admits a local modulus of continuity μ *with linear average* $\int_0^1 \mu(x,n)\,dx \leq n + \mathcal{O}(1)$.

b) *More generally the d-fold iterate* $f^{(d)} = f^{(d-1)} \circ f : [0;1] \to [0;1]$ *is computable in time an exponential tower of height d but not faster in the worst case, even relative to any oracle — while still having polynomial average-time complexity and local modulus of continuity with linear average.*

c) *There exists a continuous function* $f : [0;1] \to \mathbb{R}$ *with exponential both worst-case and average-case complexity, regardless of oracle access.*

d) *The function* $F : (0;1] \times (1;\infty) \to (1;\infty)$ *defined by* $F(x,n) := x^{-1+1/n}$ *is polynomial on average, but the composition* $(x,n) \mapsto F\big(x, F(x,n)\big)$ *is not.*

Item c) may be regarded as an explicit, real, and relativizing variant of [LiVi97, Theorem 4.6.1].

Future work will extend Definition 1 from (functions over) $[0;1]$ to compact separable metric spaces equipped with fiber-compact probabilistic names in the sense of TTE [Schr04, ScSi06].

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On Computability of Navier-Stokes' Equation*

Shu Ming Sun[1], Ning Zhong[2], and Martin Ziegler[3]

[1] Virginia Tech [2] University of Cincinnati [3] TU Darmstadt

The (physical) Church-Turing Hypothesis postulates that every physical phenomenon or effect can, at least in principle, be simulated by a sufficiently powerful digital computer up to arbitrarily prescribable precision. Its validity had been challenged, though, in the rigorous framework of Recursive Analysis: there is a computable $\mathcal{C}^1$ initial condition to the Wave Equation that leads to an incomputable solution [PEZh97]. The controversy was later resolved by demonstrating that, in the both physically [Zieg09] and mathematically more appropriate Sobolev space settings, the solution is computable uniformly in the initial data [WeZh02]. Recall that functions f in a Sobolev space are not defined pointwise but by local averages in the L_q sense** (e.g. $q = 2$ corresponding to energy) with derivatives understood in the distributional sense. This led to a series of investigations on the computability of linear and nonlinear partial differential equations [WeZh05,WeZh06,WeZh07].

The (incompressible) Navier-Stokes Equation

$$\partial_t \boldsymbol{u} - \triangle \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla P = \boldsymbol{f}, \quad \nabla \cdot \boldsymbol{u} = 0, \qquad \boldsymbol{u}(0) = \boldsymbol{u}_0 \tag{1}$$

describes the motion of a viscous incompressible fluid filling a rigid box $\overline{\Omega}$. The vector field $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t) = \big(u^{(1)}, u^{(2)}, \ldots, u^{(d)}\big)$ represents the velocity of the fluid and $P = P(\boldsymbol{x}, t)$ is the scalar pressure with gradient ∇P; $\nabla \cdot \boldsymbol{u}$ denotes componentwise divergence; $\boldsymbol{u} \cdot \nabla$ means, in Cartesian coordinates, $u^{(1)}\partial_x + u^{(2)}\partial_y + \ldots$; and the function $\boldsymbol{u}_0 = \boldsymbol{u}_0(\boldsymbol{x})$ describes the initial velocity and $\boldsymbol{f}$ a given external force. Equation (1) thus constitutes a system of $d + 1$ partial differential equations for $d + 1$ functions.

The question of global existence and smoothness of its solutions is one of the Millennium Prize Problems posted by the Clay Mathematics Institute at the beginning of the 21st century; cmp. [Wieg99]. Local strong existence in time has been established, though, over various spatial L_q settings [GiMi85]. Numerical solution methods are abundant, often based on pointwise (or even uniform, rather than L_q) approximation and struggling with computational artefacts. In fact, the very last of seven open problems listed in the addendum to [PERi89] asks for a "recursion theoretic study of [...] the Navier-Stokes equation". Moreover it has been suggested [Smit03] that a hydrodynamical system could in principle be incomputable in the sense that it allows simulation of universal Turing computation and thus 'solves' the Halting problem. Indeed, recent progress towards (a negative answer to) the Millennium Problem [Tao14] proceeds by simulating a computational process in the vorticity dynamics to construct a blowup in finite time for a PDE similar to (1).

Using the rigorous framework of Recursive Analysis we approach the problem of computing, given an initial condition $\boldsymbol{u}_0$ and inhomogenity $\boldsymbol{f}$, the strong solution of (1) in the space $X_2(\Omega)$ of square-integrable vector fields $\boldsymbol{u}$ on the open unit cube $\Omega := (-1; 1)^d$ with zero boundary condition $\boldsymbol{u}\big|_{\partial\Omega} \equiv 0$ which are divergence-free (aka solenoidal) in the sense that $\triangle \cdot \boldsymbol{u} \equiv 0$ holds. We follow a common strategy used in classical existence proofs over spaces over $L_q(\Omega)$, cmp [Giga83]:

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** We use $q \in [1, \infty]$ to denote the norm index, P for the pressure field, p for polynomials, $\mathcal{P}$ for sets of (tuples of) the latter, and $\mathbb{P}$ for the Helmholtz Projection.

i) Eliminate the pressure P by (Helmholtz) projecting the equation onto the space $X_q(\Omega)$ of solenoidal (i.e. divergence-free) solutions;
ii) solve the associated linear equation for the Stokes operator $\mathbb{A} := -\mathbb{P}\triangle$ using semigroup methods and spectral estimates;
iii) extend (ii) to incorporate the nonlinearity, e.g. using an iterative approximation/Banach fixed point argument;
iv) recover the pressure by solving a Poisson Problem.

The present work effectivizes steps (i) and (ii): We introduce a natural representation (in the sense of Weihrauch's TTE) of the space $X_q(\Omega)$ of solenoidal L_q functions on $\Omega = (-1;1)^d$ with zero boundary conditions; and derive a representation for all (total but not necessarily bounded) linear operators on this space. We employ spectral analysis in order to establish that the Stokes operator gives rise to a computable semigroup; hence showing that the solution to the Stokes Dirichlet problem is uniformly computable. We also show that the Helmholtz Projection $\mathbb{P} : \big(L_2(\Omega)\big)^2 \to X_2(\Omega)$ is computable.

Future endeavors will cover the remaining two steps, thus proving local computability of strong solutions; and extend to more general domains and boundary conditions.

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The Weihrauch degrees define monads

Kazuto Yoshimura

Japan Advanced Institute of Science and Technology
k.yoshimura@jaist.ac.jp

A. Bauer and K. Yoshimura reported a logical formulation for *the Weihrauch reducibility* in the last CCA conference [1]. This research is its continuation which shows how we can develop the theory of the Weihrauch lattice based on our formulation. Let us start with introducing the formulation first.

We work on an arbitrary logic enriched type theory build on a simply typed λ-calculus [3]. The objects of our reducibility are syntactic expressions of the form $\varphi = (x, \sigma, \varphi)$ where x, σ and φ are used as metavariables which run over variables, types and formulae, respectively. Given two of such expressions, say $\varphi = (x, \sigma, \varphi)$ and $\psi = (y, \tau, \psi)$, *the instance reduction* of φ to ψ is defined to be the formula $\forall x : \sigma. \exists y : \tau. (\psi \to \varphi)$, abbreviated as $\varphi \leq_I \psi$. Collect all φ's with being well-typed of φ in the type context $x : \sigma$, and associate the degree structure of the derivabilities of instance reductions. An immediate observation is that the resulting structure forms a bounded distributive lattice; which serves the reason of referring to the structure as *an instance lattice.*

If we instantiate the above construction to *the internal theory* of Rep, the category of represented spaces and computable functions, the instance lattice is an extension of the Weihrauch lattice, and the Weihrauch degrees are captured as *double negation dense degrees*, degrees of those φ's such that $\forall x : \sigma. \neg\neg\varphi$ is derivable. Intuitively, each φ is regarded as a malti-valued function which assigns, for each point a in the represented space obtained from σ, the computational content of $\varphi[a/x]$. In the following we refer to double negation dense degrees simply as Weihrauch degrees.

We settle one fundamental question. *What is the connection between the Weihrauch degrees and the extentions of* Rep*?* There are several effectivity concepts which are defined as reducibilities to Weihrauch degrees, such as *weak computability* which is defined as reducibility to the countable parallelization of LLPO [2]. Those effectivity concepts naturally give extensions of Rep; namely the category of represented spaces and *effective functions* with respect to each effectivity concept. Some natural questions arise: *"What kind of categorical structures are observed in such extensions of* Rep*?"*, *"How much varieties of extensions of* Rep *is obtained by such simple constructions?"*, ...etc. The above fundamental question is on the top of them.

Another story comes here. *Moggi's metalanguage* is known to be an interface to talk about *Kleisli categories* whose semantics is given by pairs of Cartesian closed categories and *(strong) monads* [4]. This well-known mechanics is broadly used in context of functional programming languages, and lead us to the approach of restricting our attention to the Kleisli categories of monads on

Rep. Now the question is *"How can we connect the Weihrauch degrees and the monads on* Rep*?"*.

On the one hand, we need specific kind of Weihrauch degrees. Notice that for two Weihrauch degrees φ and ψ, the expression $\varphi/\psi = (x, \sigma, \exists y\!:\!\tau.(\psi \to \varphi))$ is again a Weihrauch degree whenever ψ is *inhabited* i.e. $\exists y\!:\!\tau.\psi$ is derivable. Say ψ is *recursion-stable* if and only if the formula $(\varphi/\psi) \leq_I \psi \to \varphi \leq_I \psi$ is derivable for any φ. A recursion-stable ψ is said to be *an effectivity degree* provided that it is inhabited. Denote by $\mathfrak{E}$ the restriction of the Weihrauch lattice $\mathfrak{W}$ to effectivity degrees. The following adjunction arises where ι denotes the identical embedding; thus $\mathfrak{E}$ is a reflective subcategory of $\mathfrak{W}$.

$$\mathfrak{E} \underset{\longleftarrow}{\overset{\iota}{\hookrightarrow}} \mathfrak{W} \quad (\top)$$

On the other hand, we need specific kind of monads. A monad $\mathscr{T} = (\mathscr{T}, \mu, \eta)$ on Rep is said to be *bimorphic* provided that its unit η is epic and monic in the functor category. A bimophic $\mathscr{T}$ is said to be *an effectivity monad* proveded that the Weihrauch degrees of the components of its unit have a join in the Weihrauch lattice. For two effectivity monads, there is at most unique morphism from one to the other; hence the existence of such at most unique morphism defines a reducibility notion. Denote by $\mathfrak{E}^*$ the induced degree structure of effectivity monads. An adjunction naturally arises which is displayed below.

$$\mathfrak{E} \underset{\longleftarrow}{\overset{\varphi \mapsto {}^{\varphi}(-)}{\hookrightarrow}} \mathfrak{E}^* \quad (\top)$$

The embedding $\varphi \mapsto {}^{\varphi}(-)$ is determined by the following universal property: for an effectivity degree φ, all the components of ${}^{\varphi}(-)$'s unit are dominated by φ in the Weihrauch lattice; and if all the components of an effectivity monad $\mathscr{T}$'s unit are dominated by φ, then $\mathscr{T}$ is reducible to ${}^{\varphi}(-)$.

Based on the above two adjunctions, we investigate the structures of the Kleisli categories of monads defined by Weihrauch degrees. One point is that the Kleisli categories provide certain realizability interpretations. So we also investigate their behaviors as models of a higher order extension of Heyting arithmetic. In particular we explore a condition for forcing the Kleisli category of the monad defined by a Weihrauch degree φ to validate the very φ.

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